Proving Termination with polyhedra

Laure Gonnord
with Christophe Alias, Alain Darte, and Paul Feautrier
(Compsys, ENS Lyon)

LIFL - LIP

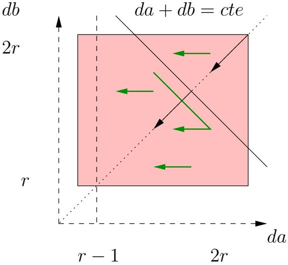
http://laure.gonnord.org/pro/ — Laure.Gonnord@lifl.fr

thx to C. Alias who gave the sources of his slides

Context : Transforming WHILE into FOR

Example : GCD of 2 polynomials

```
da = 2r; db = 2r;
while (da >= r) {
  cond = (da >= db || A[expr] == 0);
  if (!cond) {
    tmp = db; db = da; da = tmp - 1;
  } else da = da - 1;
}
```



Hard to optimize for a high-level synthesis tool :

- No loop unrolling possible.
- Limited software pipelining.

▶ Need to bound the number of iterations.

Contributions

Program termination with global multi-dimensional affine rankings

- Proven to be complete, fully implemented
- **Worst-case computational complexity**, in case of success.
- Sometimes a candidate to be an infinite loop
- WIP : sufficient preconditions for termination.

▶ for **general** flowcharts programs (with a proper preprocessing !)

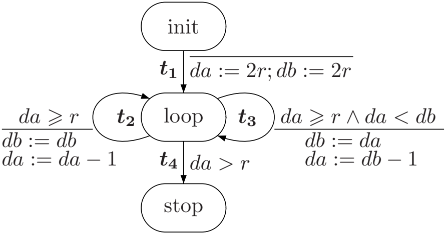
1 The Method

2 Implementation and Experimental results

3 Extensions and work in progress

From a Program to an affine Automaton

```
// expression expr,
// array A,
// r>0 integer.
  da = 2r; db = 2r;
  while (da >= r) {
    cond = ( da >= db ||
      A[expr] == 0 );
    if (!cond) {
      tmp = db;
      db = da;
      da = tmp - 1;
    }
    else da = da - 1;
  }
```

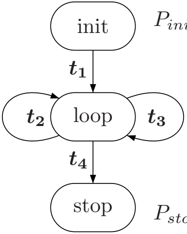


- Safe abstractions of non-affine behaviours.

From an affine Automaton to invariants

System of equations :

$$
\begin{cases}
P_{init} = \{1 \leqslant r\} \\
\boldsymbol{P_{loop}} = P_{init} \cup t_2(\boldsymbol{P_{loop}}) \\
\qquad\qquad \cup t_3(\boldsymbol{P_{loop}}) \\
P_{stop} = t_4(P_{loop})
\end{cases}
$$



$P_{init} = 1 \leqslant r$

$$P_{loop} = \begin{cases} r - 1 \leqslant da \leqslant 2r \\ r \leqslant db \leqslant 2r \\ 1 \leqslant r \end{cases}$$

$$P_{stop} = \begin{cases} 1 \leqslant r \\ r \leqslant db \leqslant 2r \\ r - 1 \leqslant da < r \end{cases}$$

► Fixpoint system with affine guards and actions. Use **abstract interpretation** to get (an over approximation of) the live space of variables.

Ranking functions

A ranking function is :

- A mapping from $(state, value)$ to a well-founded set
- Decreasing (strictly) on each transition.

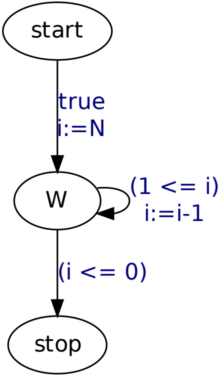
► We restrict to $\mathbb{N}^p$ with $\leqslant_{lex}$, and **multidimensional** affine rankings :

$$\rho(k, \vec{x}) = A_k.\vec{x} + \vec{b_k}$$

Finding a ranking function - 1

The 1D-case :

```
assume(N>0);
i=N;
while(i>0) --i;
```



Searching for :

$$\begin{array}{rcl}\rho(start,\vec{x}) & = & \alpha^1_{start}.\mathbf{i} + \alpha^2_{start}.\mathbf{N} \\ & + & \alpha^3_{start}.\mathbf{i_0} + \alpha^4_{start}.\mathbf{N_0} + \alpha^5_{start} \\ \rho(w,\vec{x}) & = & \alpha^1_{w}.\mathbf{i} + \dots \\ \rho(stop,\vec{x}) & = & \alpha^1_{stop}.\mathbf{i} + \dots \end{array}$$

The constraints are :

- For each pc : $\rho(pc,\vec{x}) \geqslant 0$ on P_{pc}
- For each transition $(\vec{x'} - \vec{x}) \in t \Rightarrow \rho(dest,\vec{x'}) - \rho(src,\vec{x'}) > 0$

Finding a ranking function - 2

The 1D-case (cont') : incoding into a **linear programming** problem !

1. Constraints $\rho(pc, \vec{x}) \geqslant 0$ on P_{pc} :
   - Invariant for W = $\{N_0 > 0, N = N_0, 0 \leqslant i \leqslant N\}$
   - Farkas lemma

$$\begin{array}{rcl} \rho(W, \vec{x}) & = & \lambda_W^1.(N_0 - 1) + \lambda_W^2.(N_0 - N) \\ & + & \lambda_W^3.(N - N_0) + \lambda_W^4.i + \lambda_W^3.(N - i) \end{array}$$

+affine form for $\rho(W, \vec{x})$:

$$\rho(W, \vec{x}) = \alpha_W^1.i + \alpha_W^2.N + \alpha_W^3.i_0 + \alpha_W^4.N_0 + \alpha_W^5$$

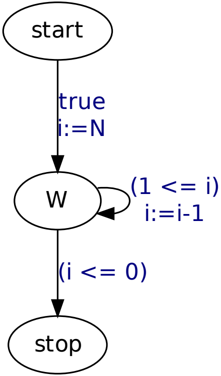
► Identifying **i** : $\alpha_W^1 = \lambda_W^4 - \lambda_W^3, \ldots$

2. Constraints for decreasing transitions : similar

Finding a ranking function - 3

The 1D-case :

```
assume(N>0);
i=N;
while(i>0) --i;
```



We find :

```
state start:
2+N__o

state W:
1+i

state stop:
0
```

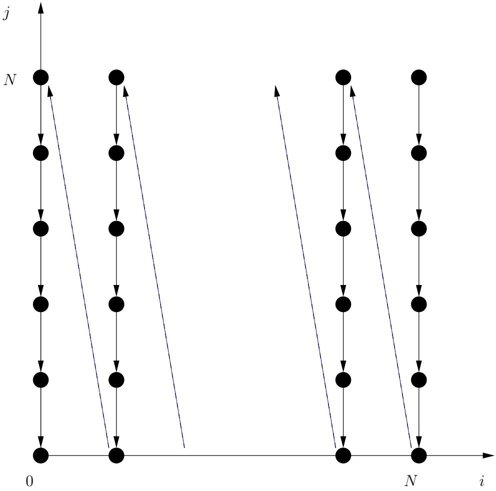
Finding a ranking function - 4

The nD-case, a **greedy algorithm**

- $i = 0$; $T = \mathcal{T}$, set of all transitions.
- While T is not empty do
  - Find a 1D affine function σ, not increasing for any transition, and decreasing for as many transitions as possible.
  - Let $\rho_i = \sigma$; $i = i + 1$; (i^{th} dimension)
  - If no transition is decreasing, **return false**.
  - Remove from T all decreasing transitions.
- $d = i$, **return true**.

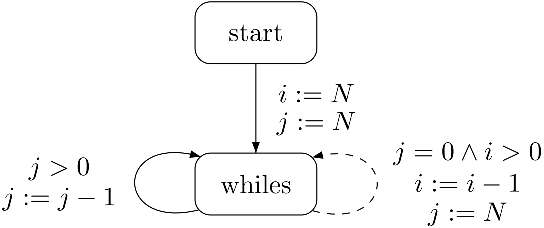
Example - 1

```
//N>0
i = N;
while(i>0)
{
   j = N;
   while(j>0) j--;
   i--;
}
```



Example - 2

```
//N>0
i = N;
while(i>0){
 j = N;
 while(j>0) j--;
 i--;
}
```

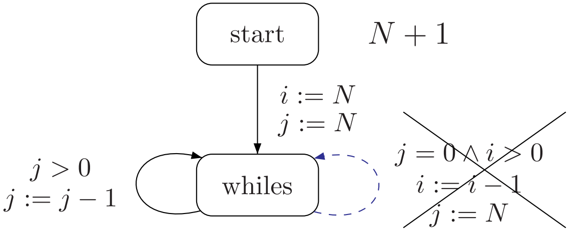


Invariant for `whiles` :

$$-\mathbf{1} < \mathbf{i} \leqslant \mathbf{N}, -\mathbf{1} < \mathbf{j} \leqslant \mathbf{N}, N > 0, N = N_o$$

Example - 2

```
//N>0
i = N;
while(i>0){
 j = N;
 while(j>0) j--;
 i--;
}
```



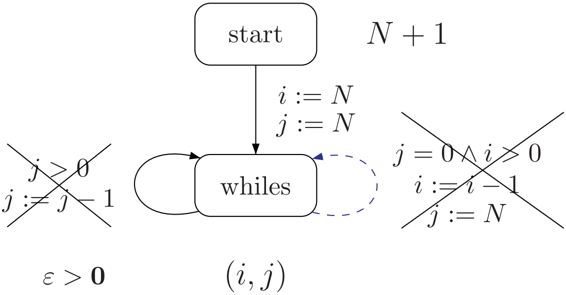
$\varepsilon = \mathbf{0}$ $(i,$ $\varepsilon > \mathbf{0}$

Invariant for `whiles` :

$$-\mathbf{1} < \mathbf{i} \leqslant \mathbf{N}, -\mathbf{1} < \mathbf{j} \leqslant \mathbf{N}, N > 0, N = N_o$$

Example - 2

```
//N>0
i = N;
while(i>0){
 j = N;
 while(j>0) j--;
 i--;
}
```

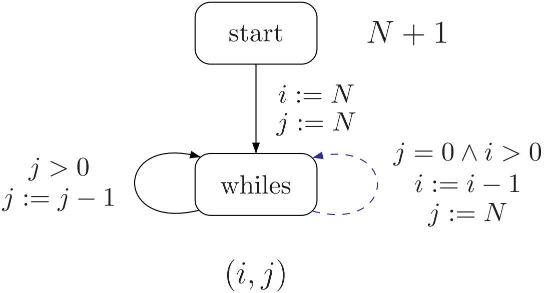


Invariant for `whiles` :

$$\color{green}{-\mathbf{1} < \mathbf{i} \leqslant \mathbf{N}, -\mathbf{1} < \mathbf{j} \leqslant \mathbf{N}}, N > 0, N = N_o$$

Example - 2

```
//N>0
i = N;
while(i>0){
 j = N;
 while(j>0) j--;
 i--;
}
```



Invariant for `whiles` :

$$\mathbf{-1 < i \leqslant N, -1 < j \leqslant N}, N > 0, N = N_o$$

In summary

From (arbitrary) flowchart programs :

- Compute an affine abstraction.
- Compute invariants on each.
- Compute and solve linear programming problems from the graph and its invariants.

An additional result !

Theorem (Completeness of greedy algorithm w.r.t. invariants)

If an affine interpreted automaton, with associated invariants, has a multi-dimensional affine ranking function, then the greedy algorithm generates one such ranking.
Moreover, the dimension of the generated ranking is minimal.

1. The Method
2. Implementation and Experimental results
3. Extensions and work in progress

Our toolsuite

1. C2FSM for the front-end
2. ASPIC for the invariants
3. RANK for the computation of the ranking function.

- The two first tools are described in :

TAPAS 2010 - Feautrier/Gonnord

Accelerated Invariant Generation for C Programs with Aspic and C2fsm

Sorting algorithms

Sorting arrays :

Name	LOCs	Time(c2fsm/analysis) [1]	dim	WCCC
selection	20	1.0/0.4	3	$\frac{N^2}{2} + \frac{3N}{2} + 1$
insertion	12	0.6/0.22	3	$\frac{N^2}{2} + \frac{3N}{2} + 1$
bubble	22	1.2/0.4	3	$N^2 + 2$
shell	23	1.0/1.1	4	$\frac{N^3}{6} - \frac{N}{6}$
heap	45	3.0/2.8	3	$4N^2 - 11N + 9$

- More examples on :

`http://www.ens-lyon.fr/LIP/COMPSYS/Tools/Ranking/`

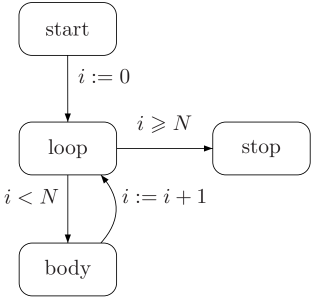
1. user time in seconds on a Pentium 2GHz with 1Gbyte RAM

Some comments on experimental results

- The algorithm scales (relatively) well.
- The form of the automaton has a strong impact on the invariants.
- The precision of invariants is **crucial**.

Piecewise-affine ranking functions

```
i=0;
while(i<N)
  {
    i = i + 1;
  }
```



We expect $loop \mapsto (1, 2(N - i))$, $body \mapsto (1, 2(N - i) - 1)$.
But... the unknown sign of N prevents to conclude.

- A piece-wise affine ranking is required :

$$
\begin{array}{rcl}
\rho(loop, i, N) & = & \left\{ \begin{array}{ll} N \geqslant 0 : & (1, 2(N - i)) \\ N < 0 : & (1) \end{array} \right. \\
\rho(body, i, N) & = & (1, 2(N - i) - 1)
\end{array}
$$

1 The Method

2 Implementation and Experimental results

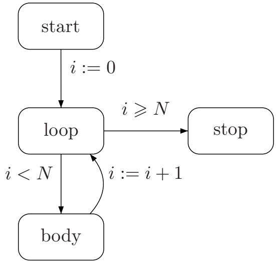
3 Extensions and work in progress

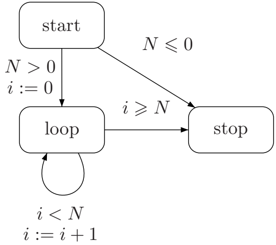
Modifying the graph : cutpoints

Definition

A set of cut points is a subset of control points such that their removal causes the graph to become acyclic

- Compute the rankings functions on the cut points after **path compression**





$(N - i)$ is found for `loop`

Computing a “WCET”

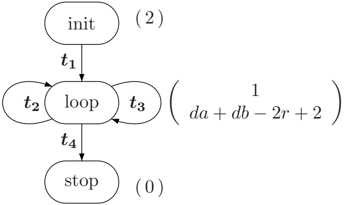
Worst-case computational complexity (WCCC) : maximum number of transitions fired by the automaton :

$$WCCC \leqslant card\Big(\bigcup_k \rho(k, P_k)\Big) \leqslant \sum_k card(\rho(k, P_k))$$

- Use **counting integer points** algorithms

$$\begin{aligned} WCCC \leqslant\ & \#\rho(\text{init}, P_{\text{init}}) \\ & +\#\rho(\text{loop}, P_{\text{loop}}) \\ & +\#\rho(\text{end}, P_{\text{end}}) \end{aligned}$$

$$= 2 + \#\{(1, i) \mid 1 \leqslant i \leqslant 2r + 2\}$$
$$= 2r + 4$$



Reference

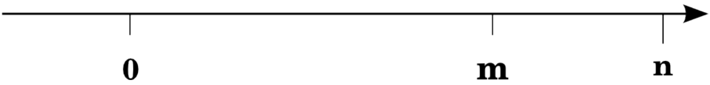
The algorithm, the extensions and the experimental results are published in

SAS 2010 - Alias/Darte/Feautrier/Gonnord

Multi-dimensional Rankings, Program Termination, and Complexity Bounds of Flowchart Programs

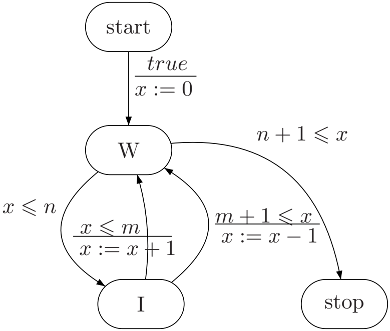
Work In Progress : finding sufficient conditions - 1

```
int catmouse(){
  int x,n,m;
  x=0;
  while(x<=n) {
    if (x<=m) ++x;
    else --x;
  }
}
```



WIP : finding sufficient conditions - 2

```
int catmouse(){
  int x,n,m;
  x=0;
  W: while(x<=n)
    {
  I:  if (x<=m) ++x;
      else --x;
    }
}
```



- Without any additional asumption, our method fails !

WIP : finding sufficient conditions - 3

1. Compute the automaton
2. Try aspic+rank ! If it fails, compute **firing conditions**
3. Find conditions on parameters to prove emptyness (**parametric linear programming**). Then retry !

- $m+1 \leq n$
  - $m+1 \leq n$ $\circ$
  - $0 \leq m$
    - $n < m+1, 0 \leq m$ $\bullet$
    - $m < 0, n < m+1$ $\bullet$

Conclusion

An algorithm to prove termination :

- on arbitrary programs
- using the link between scheduling and ranking functions
- using **polyhedra** in the large : linear relation analysis, (parametric) linear programming, computing schedules, ...
- that gives upper approximations of the worst case complexity.

Future work :

- More experiments on bigger codes : a modular approach is necessary
- Validate/Publish the sufficient conditions
- Investigate the use of disjunctive invariants