Compilation and Program Analysis (#9) : Abstract Interpretation

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Lyon 1

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Objective

Compilation vs program analysis :

- Compilation : generate code (with the “same” semantics).
- Program Analysis : infer properties, prove absence of bugs.

▶ Programs are inputs.

Inspiration for slides : M2 course Program Analysis, D. Monniaux, D. Hirschkoff.

Program analysis & Abstract Interpretation

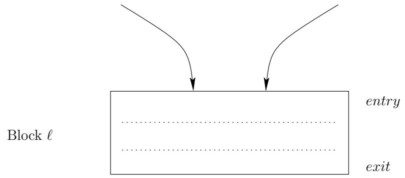
Typical questions we want to ask/bugs to avoid

- `x:=a/b` make sure that $b \neq 0$.
- `x:=t[i]` make sure that i is within the bounds of t.
- `i:=i+1` make sure there is no overflow.

or more complex properties.

Fully automatic !

Liveness

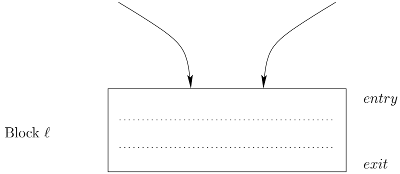


$$LV_{exit}(\ell) = \begin{cases} \emptyset & \text{if } \ell = final \\ \bigcup\{LV_{entry}(\ell') | (\ell, \ell') \in flow(G)\} \end{cases}$$

$$LV_{entry}(\ell) = \left(LV_{exit}(\ell) \backslash kill_{LV}(\ell)\right) \cup gen_{LV}(\ell)$$

- “Backward” set of recurrence equations.

Available expressions



$$AE_{entry}(\ell) = \begin{cases} \emptyset & \text{if } \ell = init \\ \bigcap\{AE_{exit}(\ell') | (\ell', \ell) \in flow(G)\} & \end{cases}$$

$$AE_{exit}(\ell) = \big(AE_{entry}(\ell) \backslash kill_{AE}(\ell)\big) \cup gen_{AE}(\ell)$$

- "Forward" set of recurrence equations.

Common points

- Computing growing sets from $\emptyset$ via *fixpoint iterations*. (or the dual)
- Sets of equations of the form (collecting semantics) :

$$\mathcal{S}(\ell) = \bigcup_{(\ell',\ell)\in E} f(\mathcal{S}(\ell'))$$

where f is computed w.r.t. the *program statements*

► $\mathcal{S}$ is an **abstract interpretation** of the program.

Concrete semantics : refreshing memories

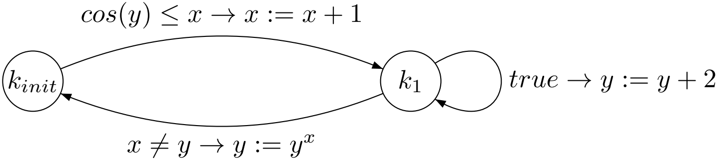
Program control points are denoted by $\ell \in \mathcal{L}$. Environnements assign values to variables.

Operational semantics :

- Recursive equations involving sets of environments.
- The fixpoint yieds a function of type $\mathcal{L} \rightarrow \mathcal{P}(\text{Var} \rightarrow \text{Val})$. (set of possible memory states of each variable at each line).

▶ Concrete semantics = least fixpoint. It exists (Knaster-Tarski's theorem). **No hope to compute it.**

Concrete semantics-revisited



Semantics of the programs as **transition systems** :

- A **state** is a pair (k, Val) :

$$\text{Val} : \text{Var} \rightarrow \mathcal{N}^d$$

  - Var is $[\![0, \ldots, d-1]\!]$ (finite set, d vars)
  - $\mathcal{N}$ is $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$

- **Initial** states : $(k_{init}, allv)$.

+ "transition relation" (concrete) denoted by $\rightarrow$.

Computing concrete semantics-reachability

Notations :

- Σ concrete states, (σ a state).
- $\Sigma_0 \subseteq \Sigma$: set of initial states.
- **reachable** states : σ is reachable iff :

$$\exists \sigma_0 \in \Sigma_0 \; \sigma_0 \rightarrow^* \sigma$$

- $R(X) = \{y \in \Sigma \mid \exists x \in X \; x \rightarrow y\}$.
- X_n : set of states reachable in at most n steps : $X_0 = \Sigma_0$, $X_1 = \Sigma_0 \cup R(\Sigma_0)$, $X_2 = \Sigma_0 \cup R(\Sigma_0) \cup R(R(\Sigma_0))$, etc.

► The sequence X_k is ascending for $\subseteq$. Its limit (= the union of all iterates) is the **set of reachable states**.

1. Come back on dataflow

2. A bit of theory
   - Computing invariants
   - Two problems to solve
   - Computing Invariants in infinite height lattices.

Computing concrete semantics-iterative computation

Remark $X_{n+1} = \phi(X_n)$ with $\phi(X) = \Sigma_0 \cup R(X)$.

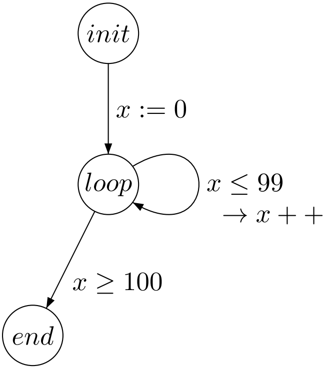
How to **compute efficiently** the X_n ? And the limit ?

- Explicit representations of X_n (list all states) : If Σ finite, X_n converges in at most $|\Sigma|$ iterations.
- else, we have to cope with two problems :
  - Representing the X_is and computing $R(X_i)$.
  - Computing the limit ?

▶ $X_\infty = \cup \phi^n(X_0)$ is the strongest **invariant** of the program

▶ Looking for overapproximations : $X_\infty \subseteq X_{result}$ also called **invariant**.

Invariants for programs



- $\{x \in \mathbf{N}, 0 \leqslant x \leqslant 100\}$ is the most precise invariant in control point `loop`.

Inductive invariants

(Inductive) invariant : set X of states s.t. $\phi(X) \subseteq X$: with

$$\phi(X) = X_0 \cup \{y \in \Sigma \mid \exists x \in X\ x \to y\}$$

Properties :

- If X et Y two invariants, then so is $X \cap Y$.
- ϕ **monotonic** for $\subseteq$ (if $X \subseteq Y$, then $\phi(X) \subseteq \phi(Y)$).
- $\phi(X \cap Y) \subseteq \phi(X) \subseteq X$, same for Y, thus $\phi(X \cap Y) \subseteq X \cap Y$.
- Same for intersections of infinitely many invariants.

▶ Thus the **strongest invariant** can be defined as the intersection of all invariants. This invariant satisfies $\phi(X) = X$, it is the **least fixed point** of ϕ.

Back to our problem

Given a program (or an interpreted automaton), find inductive invariants for each control point : Recall : a **state** is a pair (pc, Val) :

$$\text{Val} : \text{Var} \to \mathcal{N}^d$$

- We want to compute $\text{lfp}(\phi)$ with

$$\phi(X) = X_0 \cup \{y \in \Sigma \mid \exists x \in X \; x \to y\}$$

and $\to$ entails the **concrete semantics** of the program.

This is unfeasible in general

1. Come back on dataflow

2. A bit of theory
   - Computing invariants
   - Two problems to solve
   - Computing Invariants in infinite height lattices.

Representing sets of valuations

First problem to cope with : **represent sets of** valuations

$$\text{Val} : \text{Var} \rightarrow \mathcal{N}^d$$

- Var is $[\![0, \ldots, d-1]\!]$ (finite set, d vars)
- $\mathcal{N}$ is $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$

► Find a finite representation ! **abstract value/abstract domain**.

Computing R

Second problem to cope with : **computing** the transition relation

$$R(\ell, X) = \{(\ell', x') | \exists x \in X \text{ and } (\ell, x) \to (\ell', x')\}$$

- X is a (representation of a) set of valuations
- $\to$ is the program transition function (the semantics)

► We have to **adapt** $\to$ into an **abstract semantics** $\to^\sharp$. R will be changed into $R^\sharp$.

Some examples of abstractions

For numerical values, we can abstract $\mathcal{P}(\text{Var} \rightarrow \text{Val})$ by :

- Signs
- Constant+Value / Non Constant
- Intervals (Boxes)
- More complex shapes (see later).

The function used to abstract elements of $\mathcal{P}(\text{Var} \rightarrow \text{Val})$ is denoted by α and called **abstraction**.

Two examples : signs and constants

Credits : Pierre Roux for Onera.

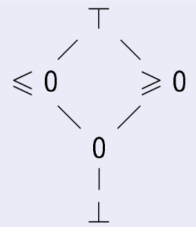
Abstraction, concretisation for signs

Abstraction : α is ?

Concretization :

$$
\begin{aligned}
\gamma(\top) &= \mathbb{Z}\\
\gamma(\leqslant 0) &=]\!]-\infty, 0]\!]\\
\gamma(\geqslant 0) &= [\![0, +\infty[\![\\
\gamma(0) &= \{0\}\\
\gamma(\bot) &= \emptyset
\end{aligned}
$$

Lattice :



Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 42 ;
while $_2$(x > 0) **{**
    $_3$x = x − 2 ;
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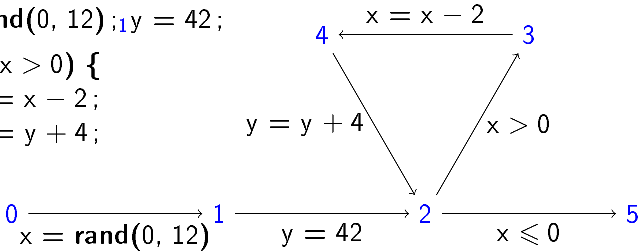

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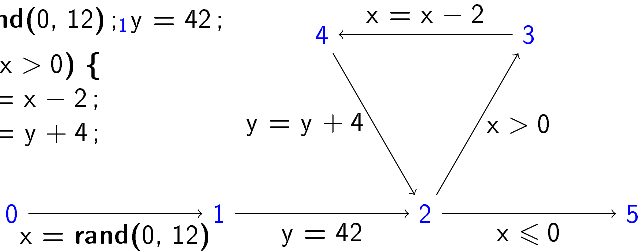

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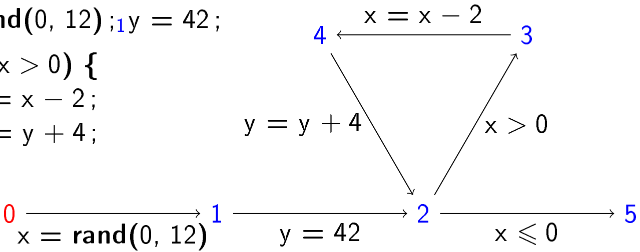

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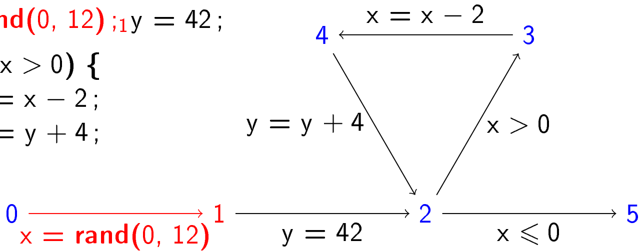

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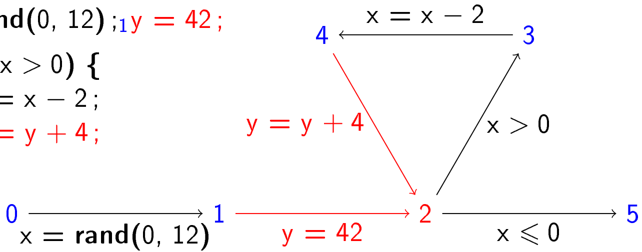

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$$(\geqslant 0, \geqslant 0) \sqcup^\sharp_{\mathrm{nr}} (\bot, \bot)$$

Exemple de calcul du point fixe abstrait

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$\mathbf{while}\ _2(\mathtt{x} > 0)\ \{$

$\quad _3\mathtt{x} = \mathtt{x} - 2\ ;$

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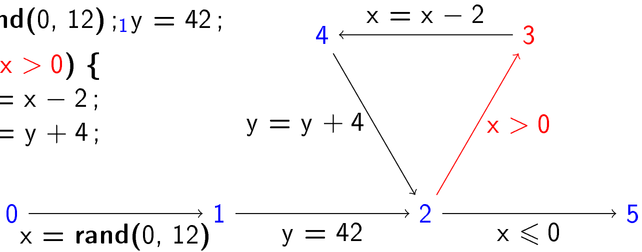

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Exemple de calcul du point fixe abstrait

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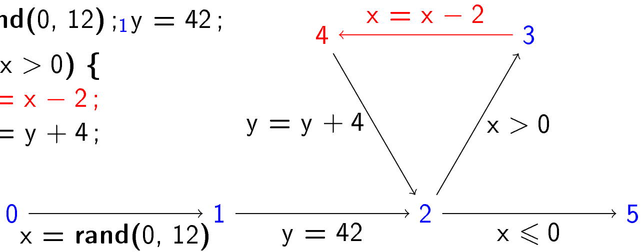

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Exemple de calcul du point fixe abstrait

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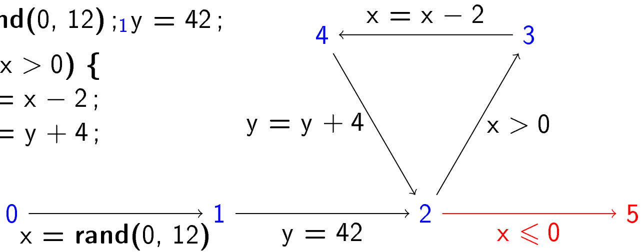

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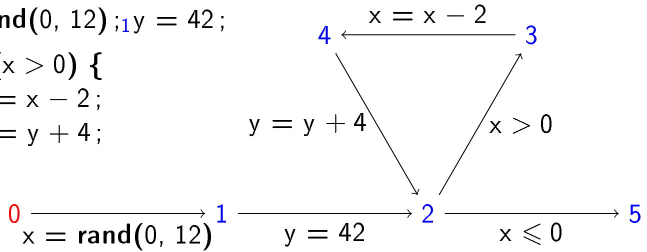

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\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$	$R_l^{\sharp\,3}$
0	$(\bot, \bot)$	$(\top, \top)$	$(\top, \top)$	
1	$(\bot, \bot)$	$(\geqslant 0, \top)$		
2	$(\bot, \bot)$	$(\geqslant 0, \geqslant 0)$		
3	$(\bot, \bot)$	$(\geqslant 0, \geqslant 0)$		
4	$(\bot, \bot)$	$(\top, \geqslant 0)$		
5	$(\bot, \bot)$	$(0, \geqslant 0)$		

Exemple de calcul du point fixe abstrait

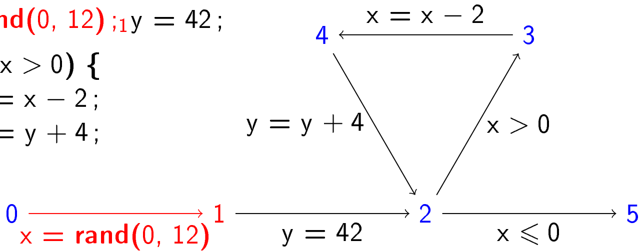
$_0$x = **rand**(0, 12) ;$_1$y = 42 ;

while $_2$(x > 0) **{**

  $_3$x = x − 2 ;

  $_4$y = y + 4 ;

}$_5$



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\begin{aligned}
R_0^{\sharp\, i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\, i+1} &= R_0^{\sharp\, i+1}\,[x \mapsto \geqslant 0] \\
R_2^{\sharp\, i+1} &= R_1^{\sharp\, i+1}\,[y \mapsto \geqslant 0] \sqcup^{\sharp}_{\mathrm{nr}} \\
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Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 42 ;
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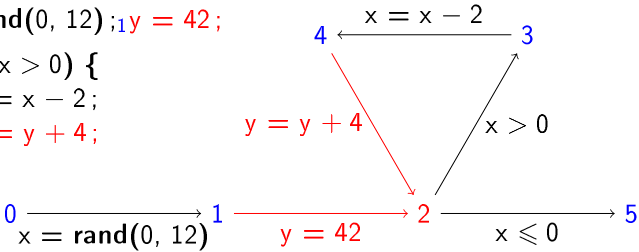

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3	$(\bot,\bot)$	$(\geqslant 0,\geqslant 0)$		
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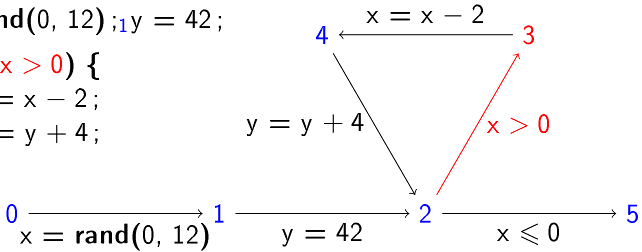
$$(\geqslant 0, \geqslant 0) \sqcup^{\sharp}_{\mathrm{nr}} (\top, \geqslant 0)$$

Exemple de calcul du point fixe abstrait

```
0x = rand(0, 12) ;1y = 42 ;
while 2(x > 0) {
    3x = x − 2 ;
    4y = y + 4 ;
}5
```



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Exemple de calcul du point fixe abstrait

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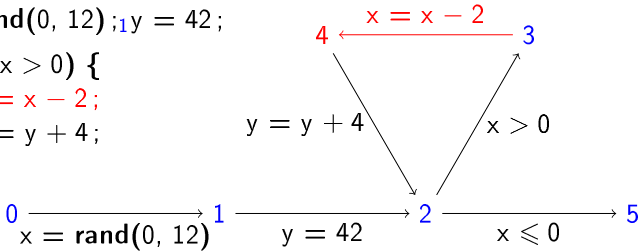

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Exemple de calcul du point fixe abstrait

${}_0$x = **rand**(0, 12) ;${}_1$y = 42 ;

while ${}_2$(x > 0) {

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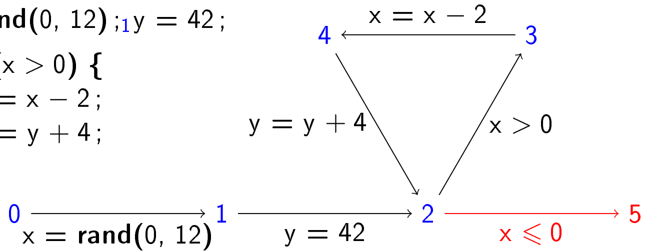
  ${}_4$y = y + 4 ;

}${}_5$



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\begin{aligned}
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Exemple de calcul du point fixe abstrait

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while $_2$(x > 0) {
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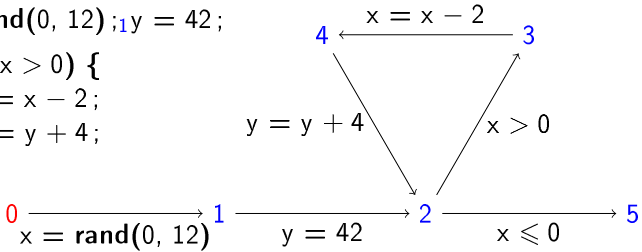

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Exemple de calcul du point fixe abstrait

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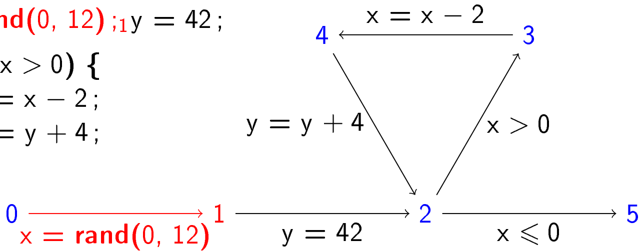

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Exemple de calcul du point fixe abstrait

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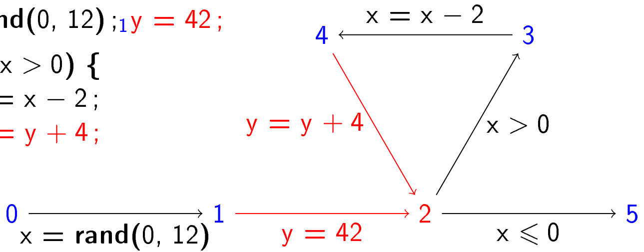

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Exemple de calcul du point fixe abstrait

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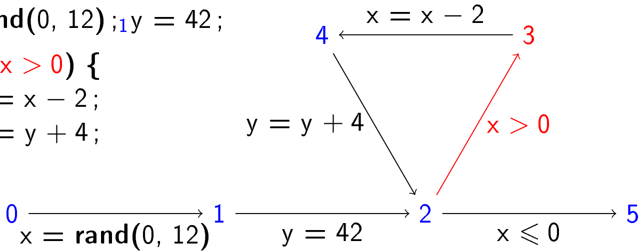

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{\color{red}R_3^{\sharp\,i+1}} &{\color{red}= R_2^{\sharp\,i+1}\left[x \mapsto R_2^{\sharp\,i+1}(x) \sqcap^\sharp \geqslant 0\right]}\\
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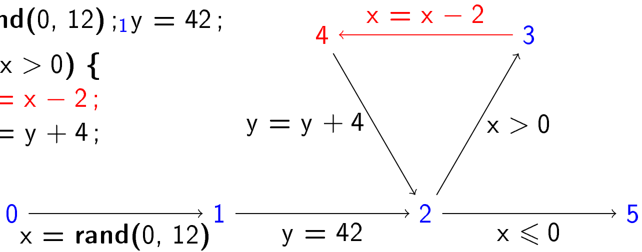

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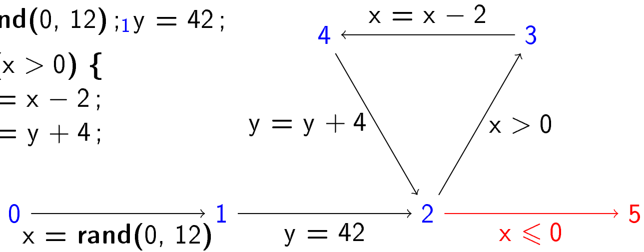

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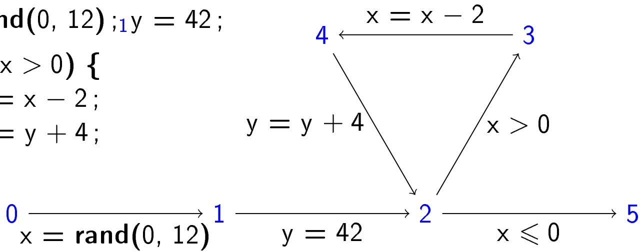
  $_4$y = y + 4 ;
}$_5$



$$
\begin{aligned}
R_0^{\sharp\, i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\, i+1} &= R_0^{\sharp\, i+1}\,[x \mapsto \geqslant 0] \\
R_2^{\sharp\, i+1} &= R_1^{\sharp\, i+1}\,[y \mapsto \geqslant 0] \sqcup^\sharp_{\mathrm{nr}} \\
&\quad R_4^{\sharp\, i}\left[y \mapsto R_4^{\sharp\, i}(y) +^\sharp (\geqslant 0)\right] \\
R_3^{\sharp\, i+1} &= R_2^{\sharp\, i+1}\left[x \mapsto R_2^{\sharp\, i+1}(x) \sqcap^\sharp \geqslant 0\right] \\
R_4^{\sharp\, i+1} &= R_3^{\sharp\, i+1}\left[x \mapsto R_3^{\sharp\, i+1}(x) -^\sharp (\geqslant 0)\right] \\
R_5^{\sharp\, i+1} &= R_2^{\sharp\, i+1}\left[x \mapsto R_2^{\sharp\, i+1} \sqcap^\sharp \leqslant 0\right]
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$	$R_l^{\sharp\,3}$
0	$(\bot,\bot)$	$(\top,\top)$	$(\top,\top)$	$(\top,\top)$
1	$(\bot,\bot)$	$(\geqslant 0,\top)$	$(\geqslant 0,\top)$	$(\geqslant 0,\top)$
2	$(\bot,\bot)$	$(\geqslant 0,\geqslant 0)$	$(\top,\geqslant 0)$	$(\top,\geqslant 0)$
3	$(\bot,\bot)$	$(\geqslant 0,\geqslant 0)$	$(\geqslant 0,\geqslant 0)$	$(\geqslant 0,\geqslant 0)$
4	$(\bot,\bot)$	$(\top,\geqslant 0)$	$(\top,\geqslant 0)$	$(\top,\geqslant 0)$
5	$(\bot,\bot)$	$(0,\geqslant 0)$	$(\leqslant 0,\geqslant 0)$	$(\leqslant 0,\geqslant 0)$

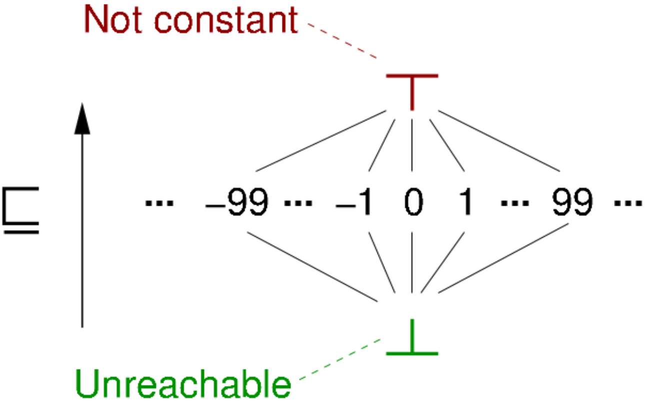
On a atteint le point fixe !

Abstraction, concretisation for constants

Abstraction : α is ?

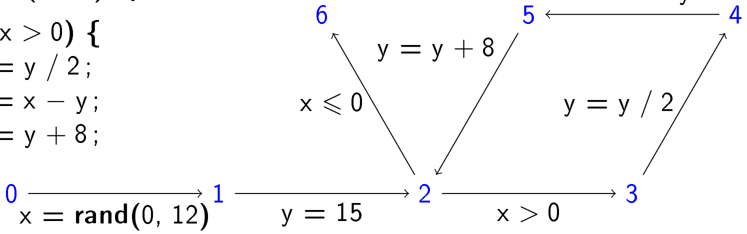
Concretization : γ is ?

Lattice :



Exemple de calcul du point fixe abstrait

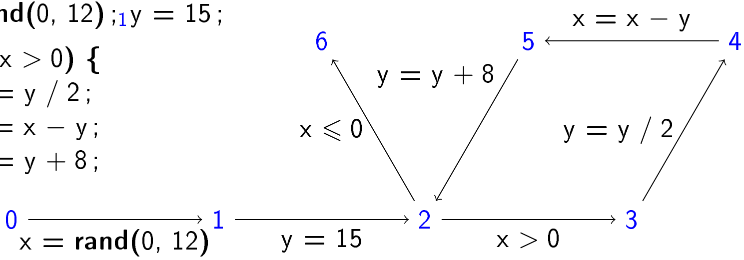
$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) {
    $_3$y = y / 2 ;
    $_4$x = x − y ;
    $_5$y = y + 8 ;
}$_6$



$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) {
    $_3$y = y / 2 ;
    $_4$x = x − y ;
    $_5$y = y + 8 ;
}$_6$

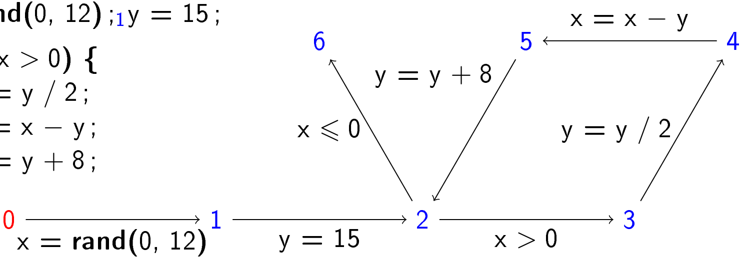


$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
& \quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot, \bot)$		
1	$(\bot, \bot)$		
2	$(\bot, \bot)$		
3	$(\bot, \bot)$		
4	$(\bot, \bot)$		
5	$(\bot, \bot)$		
6	$(\bot, \bot)$		

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) {
  $_3$y = y / 2 ;
  $_4$x = x − y ;
  $_5$y = y + 8 ;
}$_6$

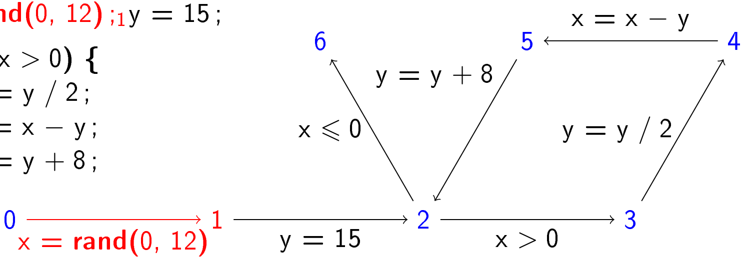


$$
\begin{aligned}
R_0^{\sharp\, i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\, i+1} &= R_0^{\sharp\, i+1}\,[x \mapsto \top] \\
R_2^{\sharp\, i+1} &= R_1^{\sharp\, i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\, i}\left[y \mapsto R_5^{\sharp\, i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\, i+1} &= R_2^{\sharp\, i+1} \\
R_4^{\sharp\, i+1} &= R_3^{\sharp\, i+1}\left[y \mapsto R_3^{\sharp\, i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\, i+1} &= R_4^{\sharp\, i+1}\left[x \mapsto R_4^{\sharp\, i+1}(x) -^{\sharp} R_4^{\sharp\, i+1}(y)\right] \\
R_6^{\sharp\, i+1} &= R_2^{\sharp\, i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot, \bot)$	$(\top, \top)$	
1	$(\bot, \bot)$		
2	$(\bot, \bot)$		
3	$(\bot, \bot)$		
4	$(\bot, \bot)$		
5	$(\bot, \bot)$		
6	$(\bot, \bot)$		

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) {
  $_3$y = y / 2 ;
  $_4$x = x − y ;
  $_5$y = y + 8 ;
}$_6$

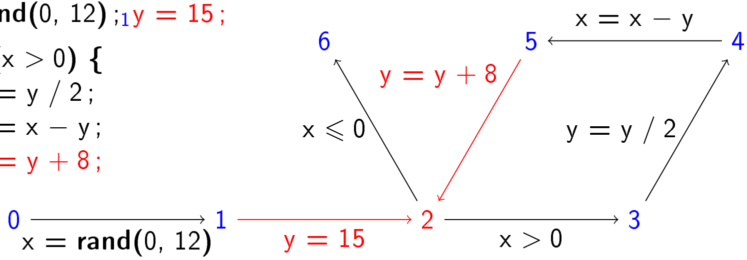


$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot,\bot)$	$(\top,\top)$	
1	$(\bot,\bot)$	$(\top,\top)$	
2	$(\bot,\bot)$		
3	$(\bot,\bot)$		
4	$(\bot,\bot)$		
5	$(\bot,\bot)$		
6	$(\bot,\bot)$		

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) {
    $_3$y = y / 2 ;
    $_4$x = x − y ;
    $_5$y = y + 8 ;
}$_6$



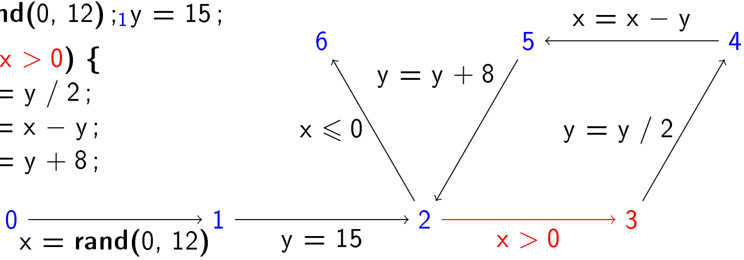
$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot,\bot)$	$(\top,\top)$	
1	$(\bot,\bot)$	$(\top,\top)$	
2	$(\bot,\bot)$	$(\top,15)$	
3	$(\bot,\bot)$		
4	$(\bot,\bot)$		
5	$(\bot,\bot)$		
6	$(\bot,\bot)$		

$$(\top, 15) \sqcup^{\sharp}_{\mathrm{nr}} (\bot, \bot)$$

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) **{**
  $_3$y = y / 2 ;
  $_4$x = x − y ;
  $_5$y = y + 8 ;
}$_6$

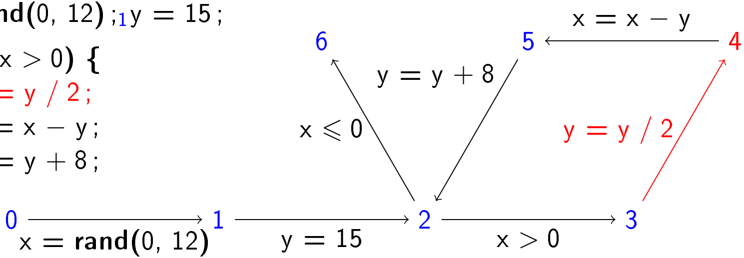


$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\qquad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot,\bot)$	$(\top,\top)$	
1	$(\bot,\bot)$	$(\top,\top)$	
2	$(\bot,\bot)$	$(\top,15)$	
3	$(\bot,\bot)$	$(\top,15)$	
4	$(\bot,\bot)$		
5	$(\bot,\bot)$		
6	$(\bot,\bot)$		

Exemple de calcul du point fixe abstrait

${}_0$x = **rand**(0, 12) ;${}_1$y = 15 ;
while ${}_2$(x > 0) {
  ${}_3$y = y / 2 ;
  ${}_4$x = x − y ;
  ${}_5$y = y + 8 ;
}${}_6$

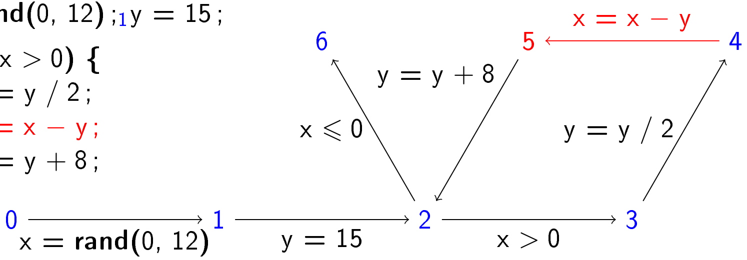


$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot, \bot)$	$(\top, \top)$	
1	$(\bot, \bot)$	$(\top, \top)$	
2	$(\bot, \bot)$	$(\top, 15)$	
3	$(\bot, \bot)$	$(\top, 15)$	
4	$(\bot, \bot)$	$(\top, 7)$	
5	$(\bot, \bot)$		
6	$(\bot, \bot)$		

Exemple de calcul du point fixe abstrait

```
₀x = rand(0, 12) ;₁y = 15 ;
while ₂(x > 0) {
    ₃y = y / 2 ;
    ₄x = x − y ;
    ₅y = y + 8 ;
}₆
```

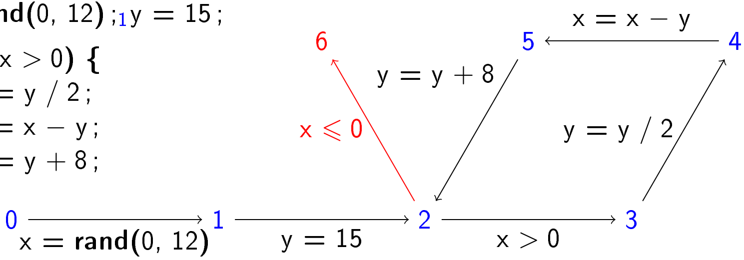


$$
\begin{aligned}
R_0^{\sharp\, i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\, i+1} &= R_0^{\sharp\, i+1}\,[x \mapsto \top] \\
R_2^{\sharp\, i+1} &= R_1^{\sharp\, i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\, i}\left[y \mapsto R_5^{\sharp\, i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\, i+1} &= R_2^{\sharp\, i+1} \\
R_4^{\sharp\, i+1} &= R_3^{\sharp\, i+1}\left[y \mapsto R_3^{\sharp\, i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\, i+1} &= R_4^{\sharp\, i+1}\left[x \mapsto R_4^{\sharp\, i+1}(x) -^{\sharp} R_4^{\sharp\, i+1}(y)\right] \\
R_6^{\sharp\, i+1} &= R_2^{\sharp\, i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot, \bot)$	$(\top, \top)$	
1	$(\bot, \bot)$	$(\top, \top)$	
2	$(\bot, \bot)$	$(\top, 15)$	
3	$(\bot, \bot)$	$(\top, 15)$	
4	$(\bot, \bot)$	$(\top, 7)$	
5	$(\bot, \bot)$	$(\top, 7)$	
6	$(\bot, \bot)$		

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) **{**
  $_3$y = y / 2 ;
  $_4$x = x − y ;
  $_5$y = y + 8 ;
}$_6$

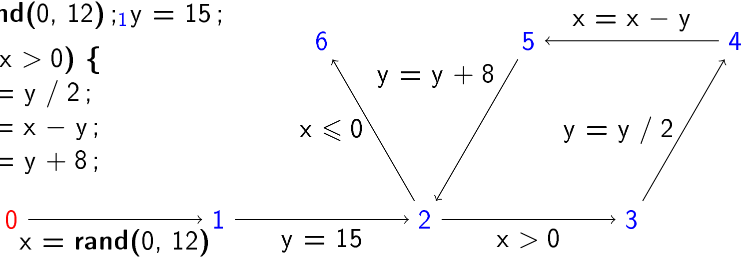


$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot, \bot)$	$(\top, \top)$	
1	$(\bot, \bot)$	$(\top, \top)$	
2	$(\bot, \bot)$	$(\top, 15)$	
3	$(\bot, \bot)$	$(\top, 15)$	
4	$(\bot, \bot)$	$(\top, 7)$	
5	$(\bot, \bot)$	$(\top, 7)$	
6	$(\bot, \bot)$	$(\top, 15)$	

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ; $_1$y = 15 ;
while $_2$(x > 0) {
  $_3$y = y / 2 ;
  $_4$x = x − y ;
  $_5$y = y + 8 ;
}$_6$

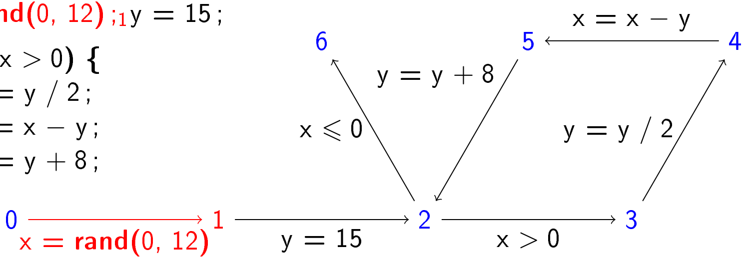


$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot, \bot)$	$(\top, \top)$	$(\top, \top)$
1	$(\bot, \bot)$	$(\top, \top)$	
2	$(\bot, \bot)$	$(\top, 15)$	
3	$(\bot, \bot)$	$(\top, 15)$	
4	$(\bot, \bot)$	$(\top, 7)$	
5	$(\bot, \bot)$	$(\top, 7)$	
6	$(\bot, \bot)$	$(\top, 15)$	

Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) {
    $_3$y = y / 2 ;
    $_4$x = x − y ;
    $_5$y = y + 8 ;
}$_6$



$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
{\color{red}R_1^{\sharp\,i+1}} &{\color{red}= R_0^{\sharp\,i+1}\,[x \mapsto \top]} \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
& \quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
R_3^{\sharp\,i+1} &= R_2^{\sharp\,i+1} \\
R_4^{\sharp\,i+1} &= R_3^{\sharp\,i+1}\left[y \mapsto R_3^{\sharp\,i+1}(y) /^{\sharp} 2\right] \\
R_5^{\sharp\,i+1} &= R_4^{\sharp\,i+1}\left[x \mapsto R_4^{\sharp\,i+1}(x) -^{\sharp} R_4^{\sharp\,i+1}(y)\right] \\
R_6^{\sharp\,i+1} &= R_2^{\sharp\,i+1}
\end{aligned}
$$

l	$R_l^{\sharp\,0}$	$R_l^{\sharp\,1}$	$R_l^{\sharp\,2}$
0	$(\bot, \bot)$	$(\top, \top)$	$(\top, \top)$
1	$(\bot, \bot)$	$(\top, \top)$	$(\top, \top)$
2	$(\bot, \bot)$	$(\top, 15)$	
3	$(\bot, \bot)$	$(\top, 15)$	
4	$(\bot, \bot)$	$(\top, 7)$	
5	$(\bot, \bot)$	$(\top, 7)$	
6	$(\bot, \bot)$	$(\top, 15)$	

Exemple de calcul du point fixe abstrait

$_0\mathrm{x} = \textbf{rand}(0,\ 12)\ ;_1\mathrm{y} = 15\ ;$

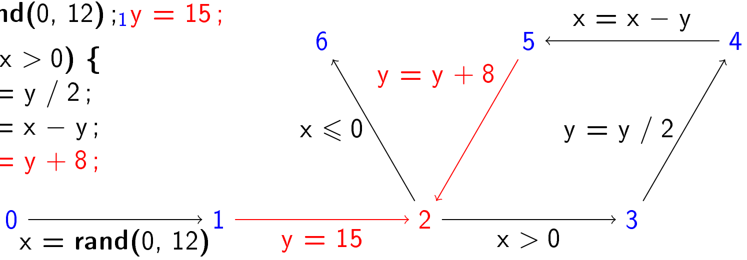
$\textbf{while}\ _2(\mathrm{x} > 0)\ \{$

$\quad _3\mathrm{y} = \mathrm{y}\ /\ 2\ ;$

$\quad _4\mathrm{x} = \mathrm{x} - \mathrm{y}\ ;$

$\quad _5\mathrm{y} = \mathrm{y} + 8\ ;$

$\}_6$



$$
\begin{aligned}
R_0^{\sharp\,i+1} &= \top_{\mathrm{nr}} \\
R_1^{\sharp\,i+1} &= R_0^{\sharp\,i+1}\,[x \mapsto \top] \\
R_2^{\sharp\,i+1} &= R_1^{\sharp\,i+1}\,[y \mapsto 15] \sqcup^{\sharp}_{\mathrm{nr}} \\
&\quad R_5^{\sharp\,i}\left[y \mapsto R_5^{\sharp\,i}(y) +^{\sharp} 8\right] \\
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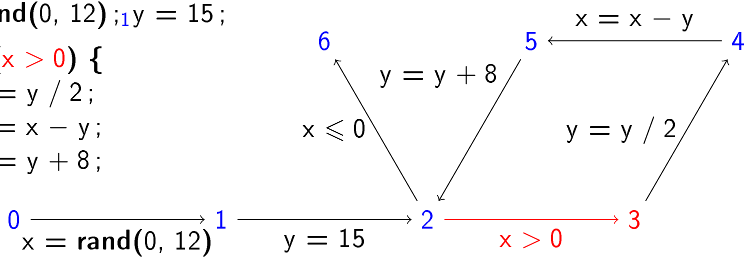
6	$(\bot, \bot)$	$(\top, 15)$	

$$(\top, 15) \sqcup^{\sharp}_{\mathrm{nr}} (\top, 7 + 8)$$

Exemple de calcul du point fixe abstrait

```
0x = rand(0, 12) ;1y = 15 ;
while 2(x > 0) {
    3y = y / 2 ;
    4x = x − y ;
    5y = y + 8 ;
}6
```



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Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;
while $_2$(x > 0) {
  $_3$y = y / 2 ;
  $_4$x = x − y ;
  $_5$y = y + 8 ;
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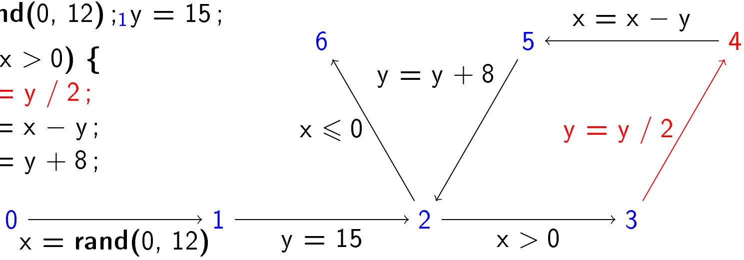

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Exemple de calcul du point fixe abstrait

$_0$x = **rand**(0, 12) ;$_1$y = 15 ;

while $_2$(x > 0) **{**

  $_3$y = y / 2 ;

  $_4$x = x − y ;

  $_5$y = y + 8 ;

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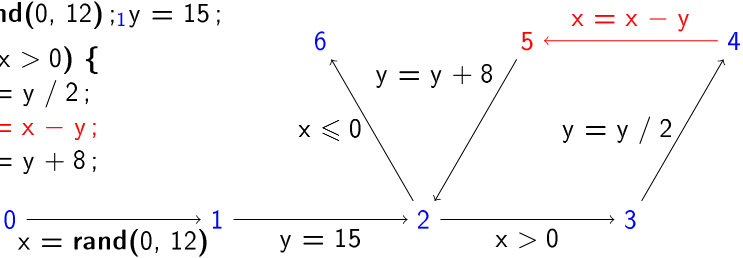

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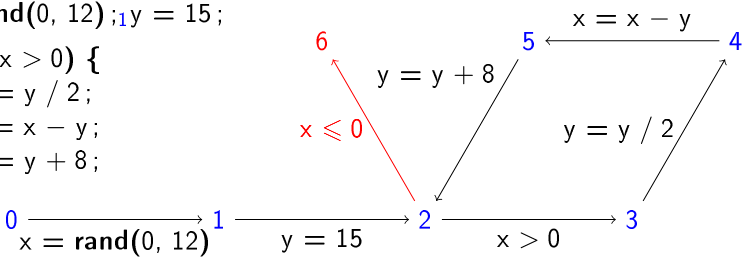

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Exemple de calcul du point fixe abstrait

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    ₄x = x − y ;
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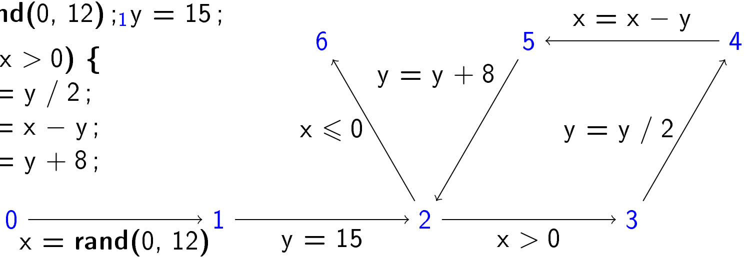
```



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On a atteint le point fixe !

Result

Termination

The fixpoint interation terminates if the underlying lattice is of finite **height**.

1. Come back on dataflow

2. A bit of theory
   - Computing invariants
   - Two problems to solve
   - Computing Invariants in infinite height lattices.

In a nutshell

The two problems also occur :

- representation of sets of valuations.
- abstract transitions (transfer functions).

there is another one, how to **terminate** !

A first example - Intervals

Try to compute an **interval** for each variable at each program point using **interval arithmetic** :

```
assume(x >= 0 && x<= 1);
assume(y >= 2 && y= 3);
assume(z >= 3 && z= 4);
t = (x+y) * z;
```

Interval for z ? $[6, 16]$

The interval lattice

Todo Picture !

An example that terminates

```
int x=0;
while (x<1000) {
  x=x+1;
}
```

Loop iterations $[0,0]$, $[0,1]$, $[0,2]$, $[0,3]$,...

How ? $\phi(X) = \mathrm{Initial\ state} \sqcup R(X)$, thus
$\phi([a,b]) = \{0\} \sqcup [a+1, \min(b,999)+1]$

- Stricly growing interval during 1000 iterations, then stabilizes : $[0,1000]$ is an **invariant**.

Termination Problem

Third problem to cope with : **stopping the computation** :

- Too many computations
- unbounded loops

One solution...

Extrapolation !

$[0,0], [0,1], [0,2], [0,3] \rightarrow [0,+\infty)$

Push interval :

```
int x=0; /* [0, 0] */
while /* [0, +infty)*/ (x<1000) {
  /* [0, 999] */
  x=x+1;
  /* [1, 1000] */
}
```

Yes ! $[0,\infty[$ is stable !

Computing inductive invariants as intervals

- Representation : intervals. The union leads to an overapproximation.
- We don't know how to compute $R(P)$ with P interval (The statements may be too complex, ...)
  ► Replace computation by simpler over-approximation $R(X) \subseteq R^\sharp(X)$.
- The convergence is ensured by **extrapolation/widening**.

► We always compute $\phi^\sharp(X)$ with : $\phi(X) \subseteq \phi^\sharp(X)$
In the end, **over-approximation** of the least fixed point of ϕ.

Computing inductive invariants as intervals - 2

(abstract) Interval operations :

- $+, -, \times$ on intervals : interval arithmetic
- union : $[a, b] \cup [c, d]$: loosing info !
- **widening** : ($I_1 \nabla I_2$ with $I_1 \subseteq I_2$)

$$\bot \nabla I = I$$

$$[a, b] \nabla [c, d] = [\text{if } c < a \text{ then } -\infty \text{ else } a,$$

$$\text{if } d > b \text{ then } +\infty \text{ else } b]$$

The idea is to infer the dynamic of the intervals thanks to the first terms.

Computing inductive invariants as intervals - 3

The widening operator being designed, we compute ($x \subseteq F(x)$)

$$\Sigma_0, Y_1 = \Sigma_0 \nabla F(\Sigma_0), Y_2 = Y_1 \nabla F(Y_1) \dots$$

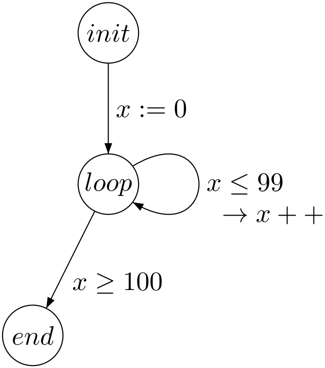
finite computation instead of : $\Sigma_0, F(\Sigma_0), F^2(\Sigma_0), \dots$ which can be infinite.

Theorem

(Cousot/Cousot 77) Iteratively computing the reachable states from the entry point with the interval operators and applying widening at entry nodes of loops converges in **a finite** *number of steps to a overapproximation of the least invariant (aka* **postfixpoint***).*

- The widening operators must satisfy the non ascending chain condition (see Cousot/Cousot 1977).

Invariants for programs - ex 1



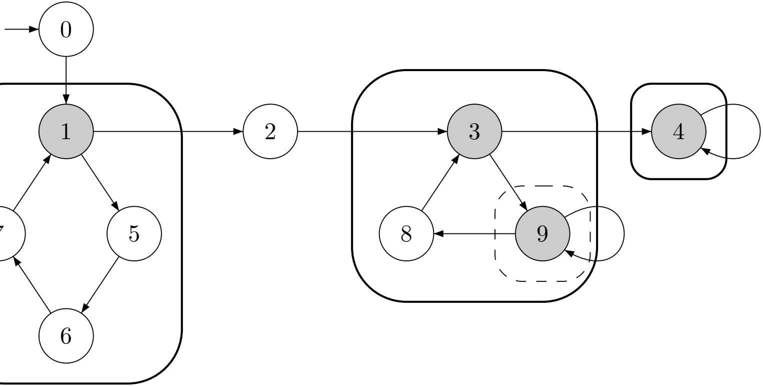
- $x \in [0, +\infty]$ in loop.

Computing inductive invariants as intervals - ex 2

```
x = random(0,7);
y = cos(x)+x
while (y<=100) {
  if (x>2) x--;
  else {
    y = -4;
    x--;
  }
}
```

Nested loops / Several loops

(Bourdoncle, 1992) Computing strongly connected subcomponents and iterate inside each :



Gray nodes are **widening nodes**

Improving precision after convergence

```
int x=0; /* [0, 0] */
while /* [0, +infty)*/ (x<1000) {
  /* [0, 999] */
  x=x+1;
  /* [1, 1000] */
}
```

we got $[0, +\infty)$ instead of $[0, 999]$. Run one more iteration of the loop : $\{0\} \sqcup [1, 1000] = [0, 1000]$. Check if $[0, 1000]$ is an inductive invariant ? **YES**

- This is called **narrowing** or descending sequence : ends when we have an inductive invariant or after k applications of the transition function.

And then ?

Plenty of ways to improve precision ▶ research problems.

Designing abstract domains

- give α, γ and union, intersection, emptyset test.
- abstract transfer functions.
- (optional) give a widening operator.