Compilation and Program Analysis (#10) : Hoare triples and shape analysis

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Inspiration D. Hirschkoff for CAP 2015-16

Hoare triples

- $\{A\}\,p\,\{B\}$ a Hoare triple

  partial correctness :

  if the initial state satisfies assertion A, and *if the execution of program* p *terminates,* then the final state satisfies assertion B
- inference rules
- expressive properties

  functional correctness rather than absence of runtime errors

Hoare logic | main ingredients

programmers

X := Y+3

(Hoare) logicians $X \geqslant Y+3$

ingredients in Hoare logic :

a language for programs p IMP

a language for assertions A

inference rules

important aspects :

- invariants in loops
- logical deduction rule
- backward reasoning (in the rule for assignment)

Hoare Logic - rules

$$\frac{}{\{A[a/X]\}\ X := a\ \{A\}} \qquad \frac{}{\{A\}\ \texttt{skip}\ \{A\}} \qquad \frac{\{A_1\}\ p_1\ \{A_2\} \qquad \{A_2\}\ p_2\ \{A_3\}}{\{A_1\}\ p_1; p_2\ \{A_3\}}$$

$$\frac{\{A \land a \geq 0\}\ p_1\ \{B\} \qquad \{A \land \neg(a \geq 0)\}\ p_2\ \{B\}}{\{A\}\ \texttt{if}\ a \geq 0\ \texttt{then}\ p_1\ \texttt{else}\ p_2\ \{B\}}$$

$$\frac{\{A_I \land a \geq 0\}\ p\ \{A_I\}}{\{A_I\}\ \texttt{while}\ a \geq 0\ \texttt{do}\ p\ \ \{A_I \land \neg(a \geq 0)\}}$$

$$\frac{A_1 \Rightarrow A_2 \qquad \{A_2\}\ p\ \{B_2\} \qquad B_2 \Rightarrow B_1}{\{A_1\}\ p\ \{B_1\}}$$

Hoare logic : metatheoretical properties 1/2

- **operational semantics and validity**
  - big step operational semantics for IMP : $(\sigma, p) \to \sigma'$
    - σ is an *environment*
    - $\sigma : \mathcal{V} \to \mathbb{Z}$ a map from variables to integers
      given some program p, σ is a partial mapping from a *finite set of variables* to $\mathbb{Z}$
  - the triple $\{A\}\, p\, \{B\}$ is **valid** :
    for all σ, if σ satisfies A and $(\sigma, p) \to \sigma'$, then σ' satisfies B

Hoare logic : metatheoretical properties 2/2

- **correctness** If the triple $\{A\}\,p\,\{B\}$ can be derived using the inference rules of Hoare logic, then it is valid.
  - NB : we could also rely on *denotational semantics*

    associate to each program p some function F_p from environments to environments
- **(relative) completeness** any valid triple can be constructed in Hoare logic, *provided* we can decide validity of the assertions *(i.e., decide whether A always holds)*
- logic rules capture the properties we want to express

Correct rules and completeness

- the 6 rules of Hoare logic are not the only correct rules
- for instance, the **rule of constancy** is correct too

$$\frac{\{A\}\,p\,\{B\}}{\{A \wedge C\}\,p\,\{B \wedge C\}} \text{ no variable in } C \text{ is modified by } p$$

- completeness : no new Hoare triple can be established if we add the rule of constancy
  - the 6 rules “tell everything”
  - using the rule of constancy makes proofs easier/more natural/more readable

somehow, completeness is not only a theoretical question

The axiom for assignment

the axiom for assignment goes backwards

$$\frac{}{\{A[a/X]\}\, X := a\, \{A\}}$$

(consider $X := X + 3$ to convince yourself)

One example

$u := 0;$

While $x > 1$ Do

if $pair(x)$ then $x := \frac{x}{2}; \ y := y * 2$ else $x := x - 1; u := u + y$

fi

od;

$y := y + u;$

Show :

$$\{x = x_0 \wedge y = y_0 \wedge x_0 > 0\} S \{y = x_0 * y_0\}$$

Programs manipulating pointers 1/2

- Hoare logic deals essentially with **control**

  $\texttt{if } a \geqslant 0 \texttt{ then } p_1 \texttt{ else } p_2 \qquad p_1; p_2 \qquad \texttt{while } a \geqslant 0 \texttt{ do } p$

- move to a *richer language :*

  add (some kind of) pointers and handling of **memory**
  - allocation
  - modification (move pointers around)
  - liberation/deallocation

Programs manipulating pointers 2/2

- different kinds of properties
  - typical runtime errors we want to detect :

    memory leaks, invalid disposal, invalid accesses

    typically, other approaches either *assume* memory safety,
    or forbid dynamic memory allocation
  - describe what programs manipulating pointers do
- adopt the same methodological framework

  Separation Logic is an enrichment of Floyd-Hoare logic

Extending Mu

- structure of memory at runtime
  - in (traditional) Hoare-Floyd logic, programs manipulate **variables**

    the **environment** just records the (integer) value of each variable

    that is all we know about the memory
  - dynamically allocated memory : add a **heap** component.

Extending the Mu language

Extanding the programming language with new constructions : nil, cons, [x]. (slides from M. Parkinson)

Example program

```
x = cons(3,3);
y = cons(4,4);
[x+1] = y;
[y+1] = x;
y = x+1;
dispose x;
y = [y];
```

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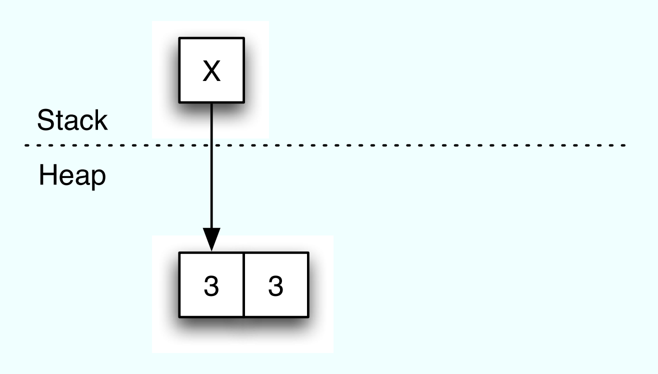

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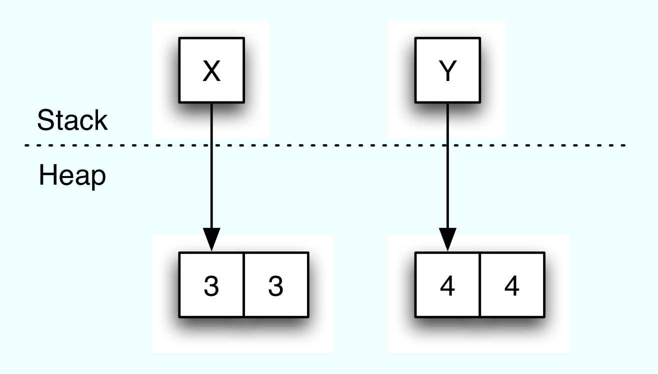

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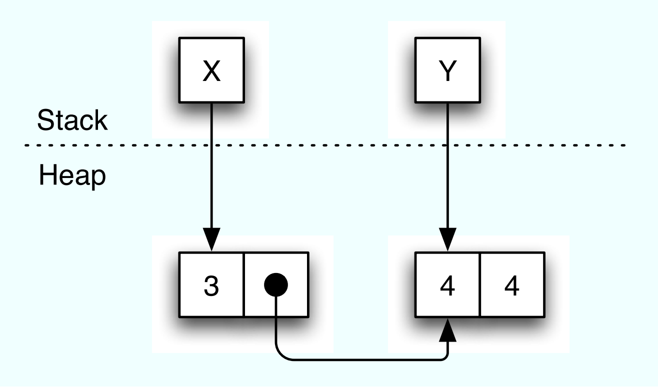

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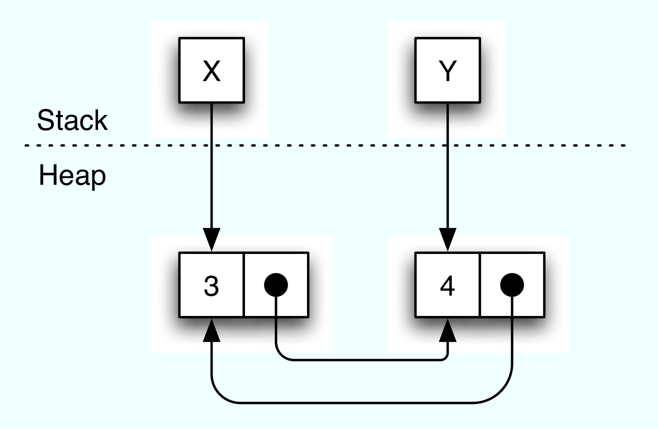

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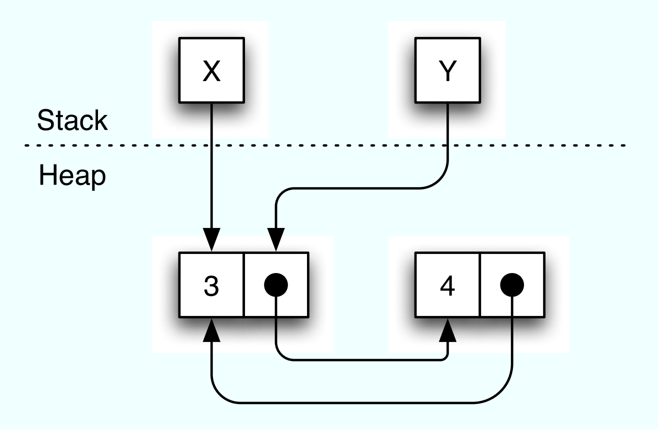

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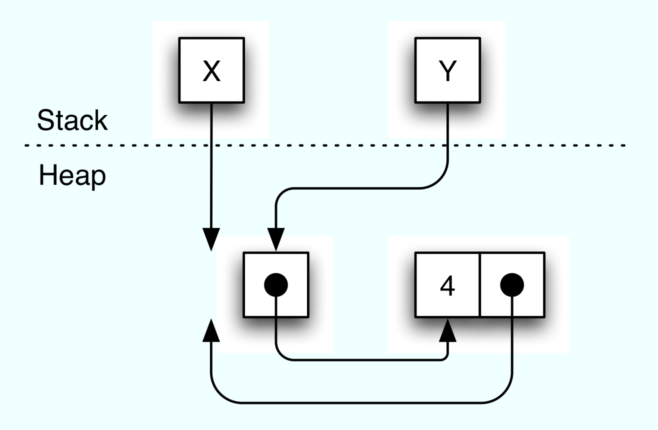

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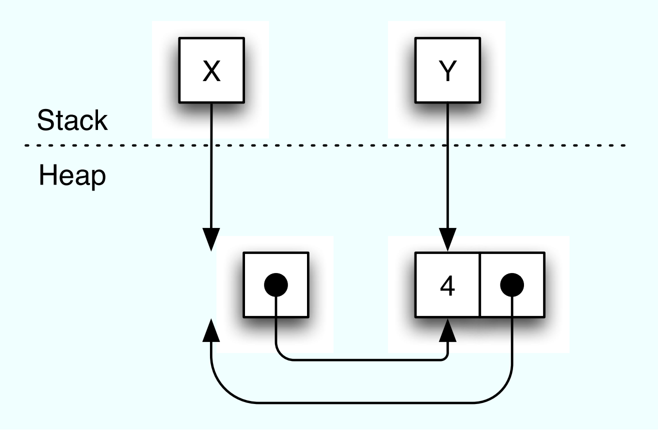


Extending Mu - example

- what does this program do ?

```
J := nil ;
while I != nil do
   K := [I + 1];
   [I + 1] := J;
   J := I;
   I := K
```

Extending the semantics 1/2

- a memory state is (σ, h) where
  - σ is a store : Variables $\rightarrow$ Values (adresses OR constants).
  - h is a heap : Adresses $\rightarrow$ Values. We denote by $dom(h)$ the set of adresses on which h is defined.
- Hoare logic assertions state properties about the environment

  $$X \geqslant Y * Z + Q \quad \wedge \quad T > 0$$

- add formulas to reason about the heap (*, $\mapsto$).
- NB : $X \mapsto 52$ usually makes more sense than $32 \mapsto 52$

(both are assertions)

Operational semantics of the new operators

Let us define $(\sigma, h) \Downarrow (\sigma', h')$ the semantics :

- Lookup : $(x := [a])$ $(\sigma, h) \Downarrow (\sigma[k/x], h)$ if $\mathcal{A}(a)\sigma = i$, $i \in dom(h)$ (else error), and $h(i) = k$.
- Mutation : $([a_1] := a_2)$ $(\sigma, h) \Downarrow (\sigma, h[k/i])$ if $\mathcal{A}(a_1)\sigma = i$, $i \in dom(h)$ (else error), and $\mathcal{A}(a_2)\sigma = k$
- Allocation : $(X = cons(a_1, \dots a_n))$ Allocation cannot fail : $\mathcal{A}(a_j)\sigma = k_j$ for all j, i is a new fresh address, then $h' = h[k_1/i, k_2/i+1 \dots]$ and $\sigma' = \sigma[x \mapsto i]$.
- Deallocation : $(free(a))$ $(\sigma, h) \Downarrow (\sigma, h \setminus i)$ $\mathcal{A}(a_2)\sigma = i$ and $i \in dom(h)$, (else error).

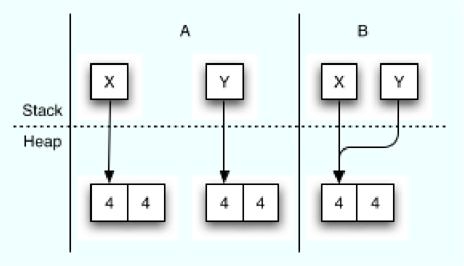
$[k/x]$ **denotes “x mapped to k”**

A new logic on states

- $(\sigma, h) \models a \geqslant 0$ iff $[\![a]\!]_\sigma = k \geqslant 0$.
- $(\sigma, h) \models \neg A$ iff not $(\sigma, h) \models A$.
- $(\sigma, h) \models A \wedge B$ iff $(\sigma, h) \models A$ and $(\sigma, h) \models B$.
- $(\sigma, h) \models \exists x, A(x)$ iff there exists $x \in \mathbb{N}$ such that $(\sigma, h) \models A(x)$
- $(\sigma, h) \models emp$ iff $dom(h) = \emptyset$.
- $(\sigma, h) \models a_1 \mapsto a_2$ iff $dom(h) = \{i\}$, $h(i) = k$ where $[\![a_1]\!]_\sigma = i$ and $[\![a_2]\!]_\sigma = k$.
- $(\sigma, h) \models A_1 * A_2$ iff $h = h_1 \biguplus h_2$ and $(\sigma, h_i) \models A_i$.

Here $\mapsto$ is a new (logic) symbol

Example of heaps



	A	B
$x \mapsto 4, 4$	✗	✓
$x \hookrightarrow 4, 4$	✓	✓
$x \mapsto 4, 4 \ * \ y \mapsto 4, 4$	✓	✗
$x \mapsto 4, 4 \ \wedge \ y \mapsto 4, 4$	✗	✓
$x \hookrightarrow 4, 4 \ \wedge \ y \hookrightarrow 4, 4$	✓	✓

where $E \mapsto E_0, \ldots, E_n \stackrel{\text{def}}{=} E \mapsto E_0 \ * \ E+1 \mapsto E_1 \ * \ \ldots E+n \mapsto E_n$

and $E \hookrightarrow E' \stackrel{\text{def}}{=} E \mapsto E' \ * \ \mathrm{true}$

Hoare triples in Separation logic | interpretation

$\{A\}\, p\, \{B\}$ holds iff

$\forall \sigma, h.$, if $(\sigma, h) \models A$, *((σ, h) satisfies A)*

then

- $(\sigma, h), p \not\Downarrow \texttt{error}$, and
- if $(\sigma, h), p \Downarrow (\sigma', h')$, then $(\sigma', h') \models B$

like in traditional Hoare logic, but :

- the state has a heap component
- absence of forbidden access to the memory

Small axioms (Hoare Triples)

- Lookup : $\{a \mapsto i \wedge X = j\} X := [a] \{X = i \wedge a[j/X] \mapsto i\}$.
  If X is not in $vars(a)$, this rule becomes
  $\{a \mapsto i\} X := [a] \{X = i \wedge a \mapsto i\}$.
- Mutation : $\{\exists i, a_1 \mapsto i\} [a_1] := a_2 \{a_1 \mapsto a_2\}$.
- Allocation : $\{X = i \wedge emp\} X := cons(a_1, \ldots a_n) \{X \mapsto a_1[i/X] * X + 1 \mapsto a_2[i/X] * \ldots * X + n - 1 \mapsto a_n[i/X]\}$
- Desallocation : $\{a \mapsto -\} free(a) \{emp\}$

► axioms for heap-accessing operations are **tight**, i.e. they only refer to the part of the heap they need to access.

About thightness

Being tight tells us the following :

- suppose we can prove $\{10 \mapsto 32\}\, p\, \{10 \mapsto 52\}$ whatever p is
- then we know that

  if we run p in a state where cell 11 is allocated, then p will not change the value of 11

The frame rule

- the rules of Hoare logic remain sound
- the rule of consistency $\dfrac{\{A\}\,p\,\{B\}}{\{A \wedge C\}\,p\,\{B \wedge C\}}$ no variable in C is modified by p

  becomes unsound

$$\frac{\{x \mapsto \_\}\,[x] := 4\,\{x \mapsto 4\}}{\{x \mapsto \_ \wedge y \mapsto 3\}\,[x] := 4\,\{x \mapsto 4 \wedge y \mapsto 3\}}$$

what if $x = y$?

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what if $x = y$?

- **the Frame Rule**

  $\dfrac{\{A\}\,p\,\{B\}}{\{A * C\}\,p\,\{B * C\}}$ no variable in C is modified by p

- separation logic is inherently **modular**

  as opposed to *whole program verification*

Separation logic : sum up

- inference rules
  - those of Hoare logic — control
  - those for the new programming constructs — memory
- important things :
  - invariants in while loops, backward rule for assignment, consequence rule
  - (tight) small axioms, footprint, frame rule
- metatheoretical properties
  - correctness
  - completeness

Reasoning about lists 1/2

$$\boxed{12} \longrightarrow \boxed{43} \longrightarrow \boxed{27} \longrightarrow \boxed{9}$$

- a linked list in memory is something like

$$(X_1 \mapsto k_1, X_2) * (X_2 \mapsto k_2, X_3) * \cdots * (X_n \mapsto k_n, \underline{nil})$$

$(X \mapsto a, b)$ stands for $X \mapsto a * (X+1) \mapsto b$

- describe the structure using assertions :

  add the possibility to write **(recursive) equations**

$$list(i) \quad = \quad (i = \underline{nil} \wedge emp) \vee (\exists j, k.\, (i \mapsto k, j) * list(j))$$

Linked lists

The preceeding formula just specifies that we have a list in memory
We can rely on "mathematical lists" ($\texttt{[]}, k :: ks$) to provide a more informative definition

$$\begin{array}{rcl} list(\texttt{[]}, i) & = & emp \wedge i = \underline{nil} \\ list(k :: ks, i) & = & \exists j.\, (i \mapsto k, j) * list(ks, j) \end{array}$$

Recursive data structures

- we can specify similarly various kinds of data structures
- we can give a meaning to such recursive definitions using Tarski's theorem

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- an **exercise**

$$list(i) \quad = \quad (i = \underline{nil} \wedge emp) \vee (\exists j, k.\, (i \mapsto k, j) * list(j))$$

  - write the code for a while loop that deallocates a linked list,
  - and prove $\{list(X)\}\, p\, \{emp\}$, where p is your program

More : Reasoning about concurrent programs

concurrent separation logic

- shared memory, several threads
- permissions, locks, critical sections
- ownership