Informatique Fondamentale IMA S8

Cours 3: Counter automata, Turing Machines and decidable problems

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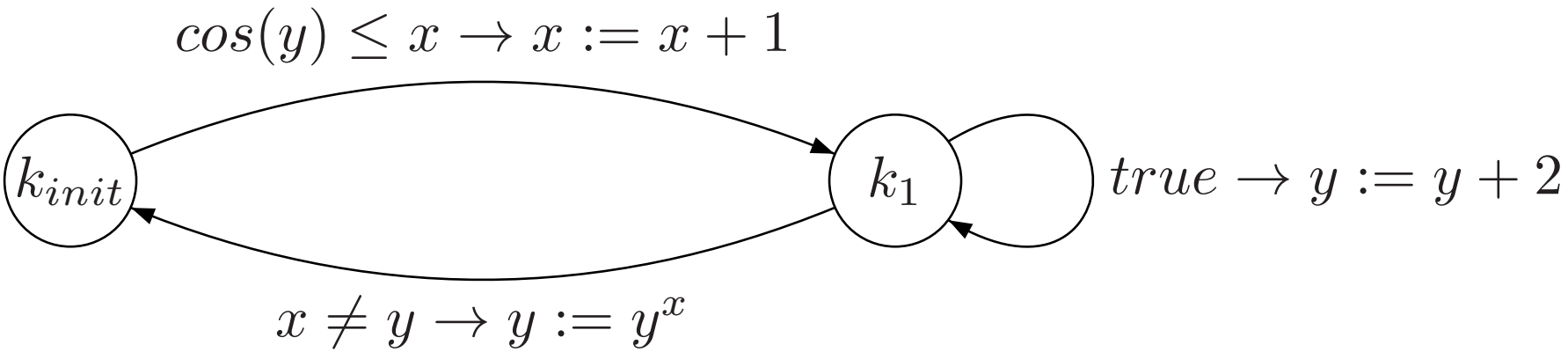
III - Counter automata, Turing Machines and decidable problems

What for ?

Express more than context-free languages !

- Give a way for automata to “**count** more” : include **variables**.

Example


$cos(y) \leq x \rightarrow x := x + 1$
k_{init}
k_1
$true \rightarrow y := y + 2$
$x \neq y \rightarrow y := y^x$

General definition

Counter Automata

- A finite number of **counters** (variables)
- A finite number of **control points**
- Transitions between them that operate on counters.

▶ The transition function is of the form $g \rightarrow a$ (g is the **guard**, a is the **action**)

How it works ? 1/3

State

A **state** is (q, σ)

- q is the current control point
- $\sigma : Var \rightarrow Val$ is a function that assigns a value (a real one or $\perp$) to all counters.

▶ Non deterministic/No notion of acceptance/Notion of **reachability**.

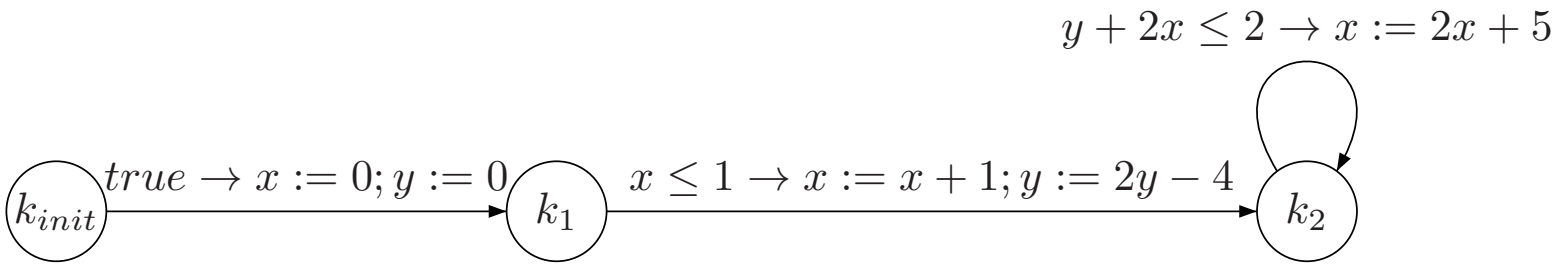
How it works ? 2/3

Given a state (q, σ), the next one is computed by the help of the transition function :

- **enabled transitions** are transition of the current control points where the current valuations of variables satisfy the guard.
- Pick one enabled transition, and compute the next state : (q', σ'). The new values for the variables are computed w.r.t. the action.

How it works ? 3/3

Example of an affine automata :

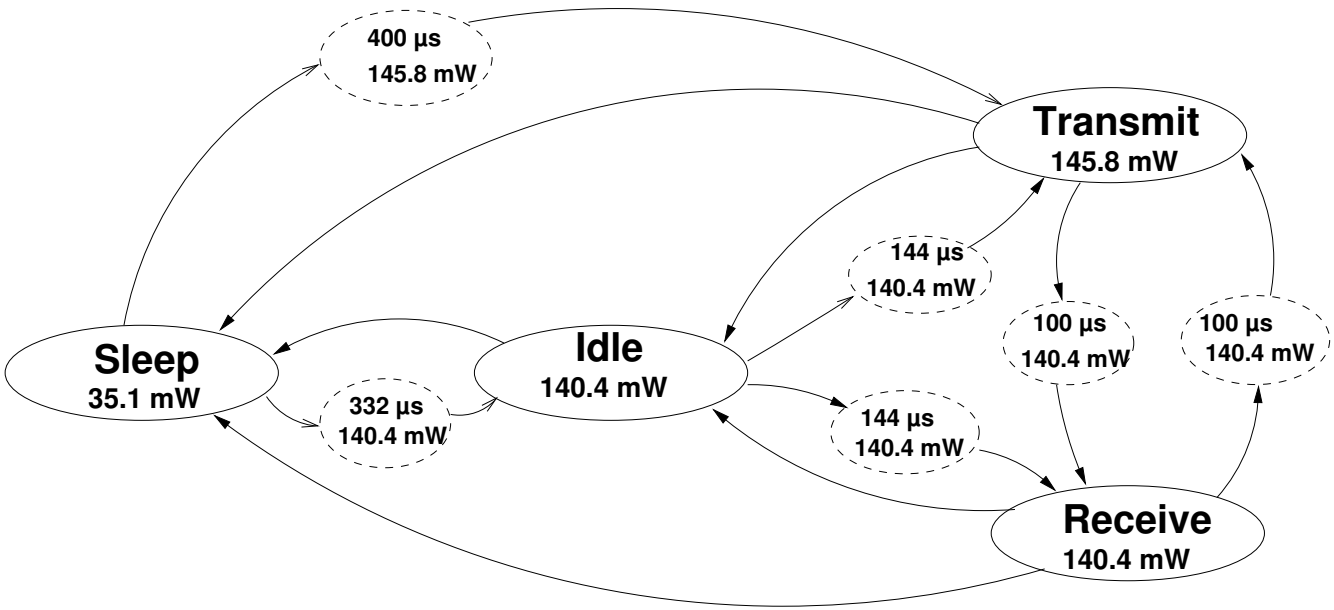


- Compute successive states.
- Exercises.

What kind of systems ?

These automata are used to encode, for example :

- Simple systems (coffee machine, ...) **specifications**
- Energy consumption of sensors :



What for ?

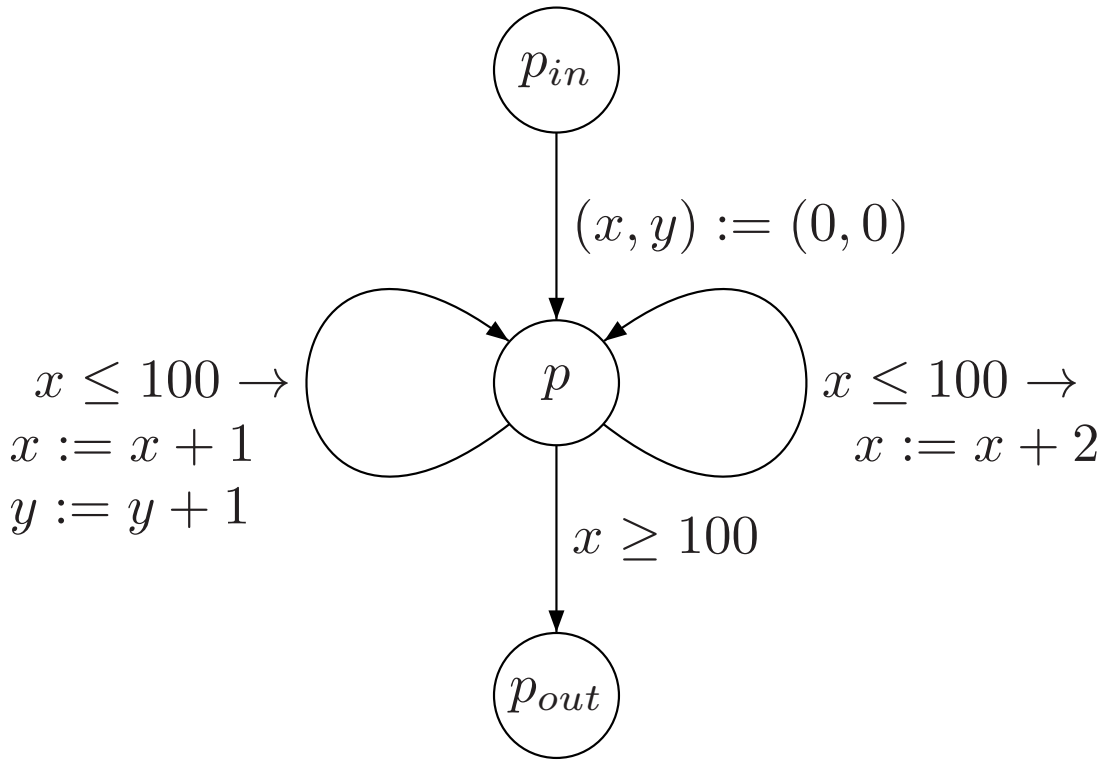
Some classical problems :

- Automatically finding invariants
- **Reachability** analysis
- Formula proving.
- (deterministic) Code generation.

Expressing programs as counter automata

An **example** :

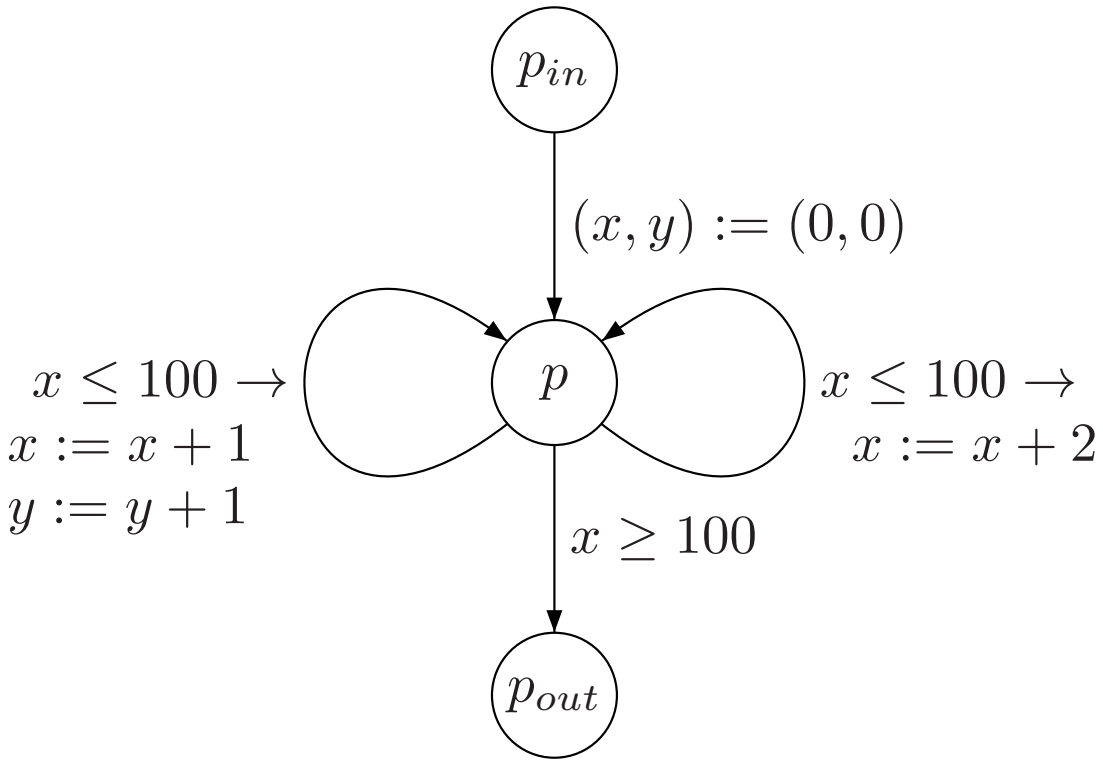
```
x:=0;y:=0
while (x<=100) do
  read(b);
  if b then
        x:=x+2
      else begin
        x:=x+1;
        y:=y+1;
      end;
  endif
endwhile
```



- Some approximations are made (arrays, Boolean, . . .)

Expressing properties of programs

Some (**safety**) properties of programs can be expressed inside the counter automaton :



- Encode the fact that $state = p_{out} \wedge y > 100$ is “bad”
- Some modifications of the automaton can be automatically done.

Expressing properties of programs - 2

Automatically deriving bad states :

```
int j;
char user[USERSZ];

for(j = 0; line[j] != EOS; ++j)
  if (!strchr("-", line[j]))
    break;

  if(j == J && line[j] == ' ') { /* long list */
    /* BUG! No bounds check. */
     assert(USERSZ>=N-j,"badstate");

     r_strcpy (user, line + j);
  }
```

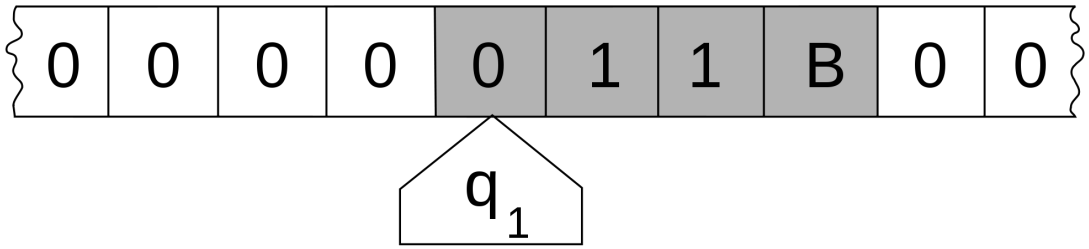
- “badstate” is reachable.

Expressing programs as counter automata

- **Pros** : finding and proving program properties
- **Cons** : how do we define the operations ? the fact that integers are encoded on bits ?

▶ There is a need for a more accurate (not specialised) model !

Turing Machine, def - 1



Turing Machine, Turing, 1935

- A **tape** (infinite in both sides) : the memory. (it has a working alphabet which contains a **blank** symbol).
- A **head** : read/write symbols on the tape.
- A finite transition table (or function)

▶ A PushDown automaton which is more flexible.

image credits : Wikipedia.org

Turing Machine, def - 2

Turing Machine elements

- States (Q), initial state (q_0), final (accepting) states (F).
- One alphabet Γ for the tape.
- $b \in \Gamma$ the blank letter.
- Transitions are finitely many : read the letter under the head, and then :
  - Erase a symbol or write one under the head
  - Move the head (read, write, or stays)
  - The “state” can be modified.

- The transition function is :

$$Q \times \Gamma \times \rightarrow Q \times \Gamma \times \{L, R, S\}$$

An Example

TM that decides if x is even : (final State : q_4)

State/char	0	1	B
q_1	(q_1,0,R)	(q_1,1,R)	(q_2,B,L)
q_2	(q_3,B,L)	(q_4,B,L)	—

Play on $BBBBBBB11BBBBB$ and $BBBBBBB10BBBBB$

- A **configuration** is a tuple (word on the tape, position of the head, state).

Adapted from : `http://www.madchat.fr/coding/algo/algo_epfl.pdf` slide 4

Another Example

Demo of a TM recognising a language : $a^n b^n c^n$

Program found here :
`http://www.cs.columbia.edu/~zeph/software/BJDweck/`

General results 1/2

There exists Turing machines for the following languages :

- palindromes
- $a^n b^n c^n$ (non algebraic language)
- a^i with i prime (non algebraic)
- a^{n^2}, with $n \geqslant 0$

▶ Turing machines are more powerful than all other models (we have seen yet)

Decidable Languages

A language that is recognised by a Turing Machine is said to be decidable.

Accepting or computing

A TM can also compute functions

TM that writes 1 if x is even, 0 else (q_6 is final state) :

State/char	0	1	B
q_1	(q_1,0,R)	(q_1,1,R)	(q_2,B,L)
q_2	(q_3,B,L)	(q_4,B,L)	—
q_3	(q_3,B,L)	(q_3,B,L)	(q_5,1,R)
q_4	(q_4,B,L)	(q_4,B,L)	(q_5,0,R)
q_5	—	—	(q_6,B,R)

Play on $BBBBBBB11BBBBB$ and $BBBBBBB10BBBBB$

Another Example

Demo of a TM computing the subtraction.

General results 2/2

There exists Turing machines for the computation of :

- $x \mapsto x + 1$
- $(x, y) \mapsto x + y$
- $x \mapsto x^2$
- all the functions you are able to write on computers

Computable functions

A function (defined for all its input) that is computable by a Turing Machine is said to be TM computable

▶ A Model of what can be computed with machines. (Church Thesis)

Decidable - Semi-decidable languages

If L is a language, L is **decidable** if there exists a Turing Machine (or an algorithm) that outputs for all w :

- 1 if $w \in L$
- else 0.

Semi-decidable :

- 1 if $w \in L$
- else does not terminate

- equivalent definition for problems.

Complexity

Link with computational complexity :

- The number of steps in the execution of a TM gives the **complexity in time**,
- The number of seen squares in the execution of a TM gives the **complexity in space**.

Examples of undecidable problems

- The halting problem for TM or counter automata (for more that 2 counters)
- Given a program, does it loops ?
- Is a given algebraic expression (with log, *, exp, sin, abs) equal to 0 ? (Richardson, 1968)
- The 10^{th} Hilbert Problem (Diophantine equations)

Summary

Turing Machines :

- A model closed to **programming languages**.
- Non deterministic.
- Acceptors for decidable languages. But some languages are not decidable !
- (equivalently) Computes solutions to problems. But ...

Important fact

Some common problems are undecidable !

Counter automata :

- are a more simpler model
- have the same power of expression as Turing Machines.
- Reachability is **undecidable** too. But we can do approximations.