A Game for the Characterization of Realizable Gauss Words.

ABSTRACT

Every diagram of knot can be represented by a Gauss word. The question if a word represents such a diagram, the realizability problem, has been solved by topological arguments (see [1]). This paper gives a combinatorial formalization of these results, using a simple game of two players. That provides a polynomial time algorithm for realizability.

1. Gauss words and Diagrams.

Definition. (words). Let n be a positive integer. G_n is the set of words on the alphabet $\{1, 2, \ldots, n\}$ where each letter occurs exactly twice. For any word A of G_n and any letter i, the word $Loop_i(A)$ is the subword between the two occurences of the letter i.

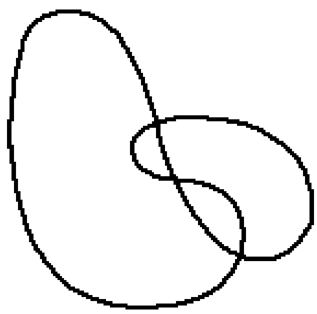
Observe that G_n contains exactly $\frac{(2n)!}{2^n}$ words and their length is $2n$.

Example. The word $A = 213123$ is in G_3. One has

$$Loop_1(A) = 3; Loop_2(A) = 131; Loop_3(A) = 12$$

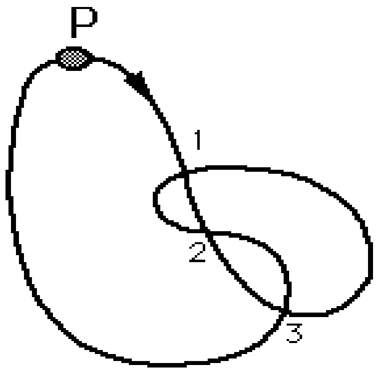
Definition. (diagram). A *diagram* is a planar close curve with finitely many transverse double points intersections. D_n is the set of diagrams with n intersections.

Example. Here is a diagram of D_3.



One can associate to every diagram of D_n, a word of G_n with the following algorithm.
First, label the n intersections of the diagram with letters in $\{1, \dots n\}$. Then fix a base point P on the diagram and an orientation. Begin with the empty word. Then follow the diagram from the base point P in the choosen orientation. At each intersection labelled by a letter i you pass, add on the right of your word the letter i. Stop your trip when you return to the base point P.

Example. The previous diagram with those labels, this base point P, and this orientation, is associated with the word 123123.



Definition. (realizable words). Say that a word A of G_n is *realizable* if there exist a diagram D of D_n, a labelling, a base point P and an orientation on D such that the word obtained using the preceding algorithm is A.

So, any diagram is associated with a finite set of words, the trivial diagram being only associated with the empty word. Conversely, there exist words wich are not realizable. For instance, if one tries to associate a diagram with the word 1212 invariably one obtains such an open curve

So 1212 is not realizable. It is obvious that the length of the subword between the two occurences of a letter i in a realizable word must be even. If it is not, as for the word 1212, the curve crosses an odd times the loop formed by the intersection i and cannot be closed. Let us call *even* any word A such that the length of every word $Loop_i(A)$ is even. The eveness of a word is a necessary condition to its realizability but is not sufficient. The word 1231245345 is even but is not realizable as we will see in the sequel.

Can one characterize combinatorialy the realizable words ? The answer is positive (see [1,2,3,4]). We give in the next paragraph a simple combinatorial version of the characterization.

2. A game on words.

Now we define a game on the words which will give a necessary and sufficient condition for the realizability of the words.

The game is played by k players on a word A of G_n. Each player has an own word which is empty at the beginning of the game. The game consists in taking the letters of A from right to left. For the first occurence of a letter, one of the k players takes this letter and appends it on the left of his word. For the second occurence, there is no choice : the player who possesses the first occurence must take the second one. If both same letters are on the left of his word, then he just cancel them, otherwise the game fails. The game is succesful if and only if all the k+1 words are empty at the end.

Example. Consider the following failing game on the word 123123 with 2 players.

WORD	PLAYER1	PLAYER2
123123		
12312	3	
1231	3	2
123	3	12
12	~~33~~	12
1		212

Observe that there are at most k^n different games on a word of G_n with k players because the first occurence of each letter can be taken by each player but the second occurence must be taken by the owner of the first one.

Definition. (players words). A word A of G_n is a k-players word if there exist a successful game on the word A with k players.

Remark. It is obvious that the k-players words are also (k+1)-players words (The new player can sleep during the game). The 1-player words of G_n are the well bracketted words with one kind of bracket associated with each letter. For instance, the word 123344256651 is a 1-player word and can be viewed as a bracketting corresponding to

$$(1(2(33)(44)2)(5(66)5)1)$$

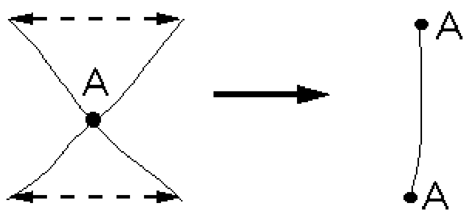
Now, we recall a well known construction on the Gauss words.

Definition. (Switch). Let A be a word of G_n and i be a letter. Denote by $Switch_i(A)$ the word A where the subword $Loop_i(A)$ is reversed. Denote by $Switch(A)$ the word obtained by the composition of $Switch_1, \ldots, Switch_n$ on the word A.

Example. Consider the word 1231245345 of G_5. One has the followings

$$\begin{aligned}
Switch_1(1231245345) &= 1321245345\\
Switch_2(1321245345) &= 1321245345\\
Switch_3(1321245345) &= 1354212345\\
Switch_4(1354212345) &= 1354321245\\
Switch_5(1354321245) &= 1354212345
\end{aligned}$$

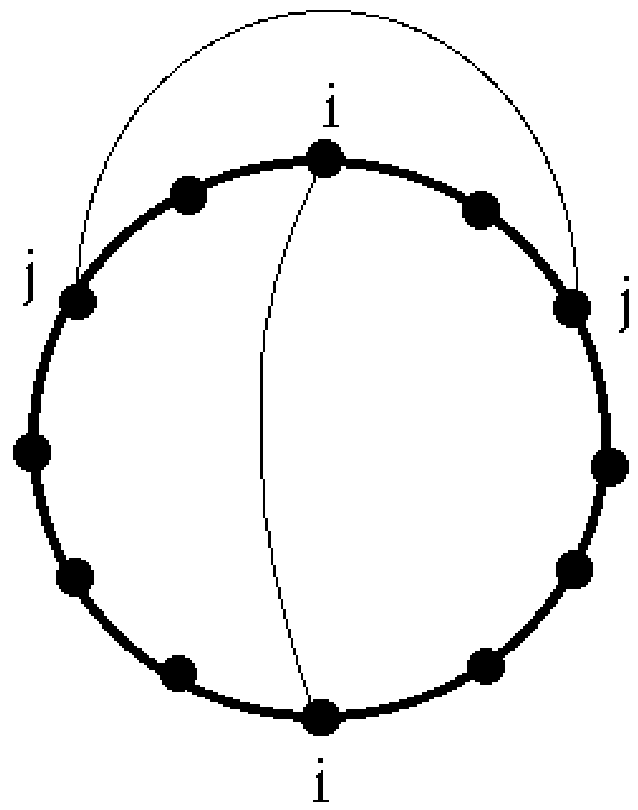
It is well known (see [1]) that the $Switch$ of a realizable word corresponds to the Jordan's curve (closed planar curve without any crossing) associated with the diagram. This operation on the words corresponds to the following topological transformation



We can now formulate the correspondance between realizability and the game defined previously with the

Proposition 1. (realizable words). *An even word A of G_n is realizable if and only if the word $Switch(A)$ is a 2-players word.*

Proof. The proposition is just a reformulation in our context of the result due to [1] that an even word is realizable if and only if one can write the letters of its $Switch$ on a circle and join both occurences of the same letter without making any crossing. That means that some letters must be joined inside the circle and the others must be joined outside as the following picture shows.



In the context of the 2 players game, the first player takes the letters joined inside the circle and the second player takes the letters joined outside.■

Example. Let us check the word 123123. The *Switch* of this word is obtained by the computation

$$Switch_1(123123) = 132123$$
$$Switch_2(132123) = 132123$$
$$Switch_3(132123) = 132123$$

And a successfull game on 132123 is

WORD	PLAYER1	PLAYER2
132123		
13212	3	
1321	23	
132	23	1
13	~~2~~23	1
1	~~3~~3	1
		~~1~~1

This characterization gives an exponential algorithm to check the realizability of any even word A of G_n. It is sufficient to check at most all the 2^n possible games of 2 players on the word $Switch(A)$. As a game is played in linear time, the algorithm is in $O(n.2^n)$. The complexity of the computation can be improved using an obvious necessary condition for a successfull game. Say that two letters i and j are *alternating* in a word A if it has the form

$$-i-j-i-j-$$

Then, the word $Loop_i(A)$ contains one and only one letter j and the letter j cannot be taken by the same player who possesses the letter i. That is obvious because if a player takes both letters i and j, as the second occurence of j comes only after the second occurence of i, the letter i cannot disapear. So the game would fail. That consideration decreases in general the number of games to be considered.

We have said previously that 1231245345 is not realizable. For a proof, one can check all the 32 possible games on its $Switch$ = 1354212345. But there is a direct argument. The letter 1, taken by the first player, is alternating with any other letter. So the letters $2, 3, 4, 5$ must be taken by the second player. But the letter 3 is alternating with 4. So there remains no possibility since there is only two players. Such argument gives in general an efficient algorithm.

3. A polynomial time algorithm for realizability.

We construct an algorithm using the alternating letters argument to decide the realizability of any word. We have said that, if two letters i and j are alternating in a word, then they cannot be assigned to the same player. Moreover, if i and k are alternating as well, the assignations of j and k must be equal, as there are only two players. To be systematic, we construct an equational system associated with a word.

Definition. (system). Let A be a word of G_n. Let $x_1, \ldots, x_n$ be formal variables. The equational system of A is the least set S containing the relations $(x_i \neq x_j)$ for i and j alternating in A and such that the followings hold for any i, j, k.
(i) $(x_i \neq x_j) \in S \Longrightarrow (x_j \neq x_i) \in S$
(ii) $(x_i = x_j) \in S \Longrightarrow (x_j = x_i) \in S$
(iii) $(x_i = x_j) \in S$ and $(x_j = x_k) \in S \Longrightarrow (x_i = x_k) \in S$
(iv) $(x_i = x_j) \in S$ and $(x_j \neq x_k) \in S \Longrightarrow (x_i \neq x_k) \in S$
(v) $(x_i \neq x_j) \in S$ and $(x_j = x_k) \in S \Longrightarrow (x_i \neq x_k) \in S$
(vi) $(x_i \neq x_j) \in S$ and $(x_j \neq x_k) \in S \Longrightarrow (x_i = x_k) \in S$
The word A is *consistent* if the set S does not contain any contradiction $(x_i \neq x_i)$.

Proposition 2. (consistency). *A word A of G_n is a 2-players word if and only if A is consistent.*

Proof. Assume that A is realizable. Then there exist a successfull game with 2 players on the word $Switch(A)$. For any letter i, denote by p_i the number of the player who took this letter during the game. Then, the assignement $x_i = p_i$ is a model of the set of equations of $Switch(A)$. For the converse, if A is consistent, then the set of equations S has a model and the relation (vi) forces any model of S to be isomorphic to $\{1, 2\}^n$, giving the number of the player who takes the letters during a successfull game.■

That gives a polynomial time algorithm to check the realizability of an arbitrary word A of G_n.

Theorem 3. *There exists an algorithm deciding the realizability of any word A of G_n in time $O(n^5)$.*

Proof. The word $B = Swich(A)$ can obviously be computed in time $O(n^2)$. Now, the realizability of A is equivalent to the consistency of B. We construct the equational system of the word B. First, the alternating pairs of letters in B can be found in time $O(n^3)$. Now we compute the minimal closure of the system. Note that it contains at most $O(n^2)$ equations $(x_i = x_j)$ or $(x_i \neq x_j)$ and any new equation can be found in time $O(n^3)$ by checking every i, j, k. So the closure can be computed in time $O(n^5)$. The consistency can be checked during this construction.■

The condition of being a 2-players word (i.e.consistent) is very simple to check algorithmically. In this way, one can obtain the tables of realizable words on G_n. Here are the cardinalities of realizable words on G_n for $n = 3, \ldots, 12$. We omit the redundances of same diagram words and composite words of G_n of the form BC with B in G_m with $0 < m < n$.

n	3	4	5	6	7	8	9	10	11	12
#	1	1	2	3	10	27	101	364	1610	7202

For instance, here are the 27 diagram words on G_8.

1231452367548678; 1231452367845678; 1231452367845876; 1231452367865478;
1231456324785687; 1231456327845876; 1231456326785487; 1231456327865874;
1231456423785687; 1231456723845678; 1231456723847658; 1231456723865478;
1231456473285687; 1231456473278658; 1231456783245678; 1231456783245876;
1231456783247658; 1231456783285674; 1231456783267854; 1231456783287654;
1234512367854678; 1234512647836578; 1234512647856378; 1234516375862748;
1234516783452678; 1234516783452876; 1234516785432876.

For the construction of the tables of realizable words, it would be useful to have a completion procedure giving all realizable words of G_{n+1} only by inserting the two occurences of the new letter $(n{+}1)$ in some realizable word of G_n. But such a construction is not always possible. For example, the word 1231452367865478 is realizable but the 8 words obtained by cancellation of any letter are not even any more and then, not realizable. So this word cannot be obtained by such a completion. Is there a systematic way to complete realizable words in order to obtain bigger ones ?

References.

[1] M. Dehn, *Über Kombinatorische Topologie*, Acta Math. **67** (1936) 123-168.

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