A caracterization of realizable Gauss words

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1 Gauss words, Gauss diagrams

In this section the notion of Gauss word is introduced, and also the problem of its realizability.

1.1 First definitions

- Let n be a positive integer. G_n is t he set of word on the alphabet $\{1, 2, \ldots n\}$ where each letter appears exactly twice. Such words are called **Gauss words**.
- The set of planar close curves with n double points is called D_n. Its elements are called **diagrams**.
- One can associate to any diagram of D_n a word of G_n :

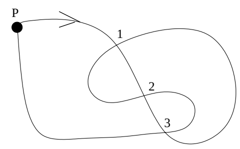

Figure 1: 123123's associated diagramm

The problem is : given a Gauss word, is it always possible to draw an associate diagram ? The answer is no : 1212 is not **"realizable"**. Thus we are led to look for a necessary and sufficient condition for the realizability of a Gauss word.

1.2 A necessary condition to be realizable

Given $A \in G_n$, we call $Loop_i(A)$ the subword of A situated between the two occurences of the letter i. Then we have a necessary condition for realizability :

$$\boxed{\text{A necessary condition for the realizability of a Gauss word } A \text{ is that the length of every } Loop_i(A) \text{ is even.}}$$

2 A game on words

In this section a game on words is described. It gives a necessary and sufficient condition for the realizability of a given Gauss word.

2.1 The game's rules

- The game is played by k player on a given word A of G_n. Every player has an own word wich is empty at the beginning of the game. The letters of A are taken one by one, reading them from right to left. One player takes a letter, and appends it at the left of its own word. If it is the first occurence of the letter, everyone can take it, but if it is the second occurence, the player who takes it must be the one who has taken the first occurence. If both same letter are on the left of the word, they are cancelled, otherwise the game fails. The game is **successful** if and only if all $k+1$ words are empty at the end.
- A word A of G_n is called a **k-player word** if there exists a successful k-player game on this word.

2.2 An important caracterization of Gauss words

Let A be a word on G_n and i be one of its letter. Denote by $Switch_i(A)$ the word A where the subword $Loop_i(A)$ is reversed, and by $Switch(A)$ the word obtained by the composition of the $Loop_i$ transformations on the word A. Then, seeing that the transformation $Switch$ is a way to see Jordan's transformation on planar curves, we obtain the new caracterisation :

An even word A of G_n is realizable iff the word $Switch(A)$ is a 2-player game.

This equivalence gives us an algorithm to check realizability : its complexity is in $O(n.2^n)$ because it checks all 2^n possible 2-player games.

2.3 An easy improvment

- If we have the following configuration in the word A :

$$-i-j-i-j-$$

  (the letters i and j are said to be **alternating** in A), then i and j *must* be taken by two different players.
- This consideration improve the last algorithm in most of the times.

3 A polynomial algorithm for realizability

This last consideration leds us to buid a system of relationships from the considered word where the alternating letters play a particular role.

3.1 The notion of consistence

- We build from the letters of A a set of relationships on formal variables $(x_i)_{1\leq i\leq n}$. The alternating letters in A play a particular role. The relationships are of type $x_i = x_j$ or $x_i \neq x_j$.
- A is said to be **consistent** if its associated set does not contain any contradiction.

Then we obtain the following caracterization :

A word A of G_n is a 2-player game iff A is consistent.

3.2 A polynomial algorithm

The last considerations gives us a polynomial algorithm to check if a Gauss word is realizable. From A we build $Switch(A)$ in polynomial time, and then the construction and the check-out of the consistence of the associated set is also polynomial.

Finally we have a polynomial algorithm to check the realizability of a Gauss word.