Colloquium

ENS Lyon - March 2001

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A game for the Characterization of Realizable Gauss Words

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PLAN :

- What's the problem with Gauss words ?
- A game on words
- A polynomial algorithm for realizability

What's the problem with Gauss Words ?

Some definitions

- Let $n > 0$. G_n is the set of words on the alphabet $\{1, 2, \ldots, n\}$ where each letter occurs exactly twice.
  ex : $A = 123123$.

- Call $Loop_i(A)$ the subword beetween the two occurrences of the letter i.
  ex : $Loop_1(A) = 23$, $Loop_2(A) = 31$, $Loop_3(A) = 12$.

Diagrams

A diagram = a planar close curve with finitely transverse double intersections points.

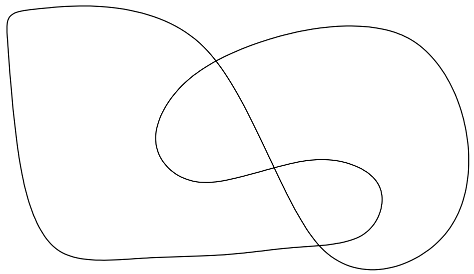

FIG. 1 – A diagram of D_3

Realizability

FIG. 2 – 123123 is "realizable" ; is 1212 realizable ?

Conditions for realizability

- $\forall i$, $|Loop_i(A)|$ must be even. A is said to be " even".
- This condition is not sufficient, see 1231245345.

A game on Words

Presentation of the rules

- An example :

WORD	PLAYER1	PLAYER 2
123123		
12312	**3**	
1231	3	**2**
123	3	**12**
12	$\cancel{3}$ $\cancel{3}$	12
1		**212**

- k^n possible games on a word of G_n
- $A \in G_n$ is a k-player word if there exists a k-player game on this word.

Switch

- $Switch_i(A)$ is the word A where $Loop_i(A)$ is reversed.
  Example : $A = 123123$, $Switch_1(A) = 132123$,
  $Switch_2(A) = 121323$, $Switch_3(A) = 123213$.

- $Switch(A)$ is the composition of the $Switch_i(A)$:
  $Switch(123123) = 132123$.

Switch and geometrical transformation

The *Switch* of a word corresponds to the repeated following transformation :

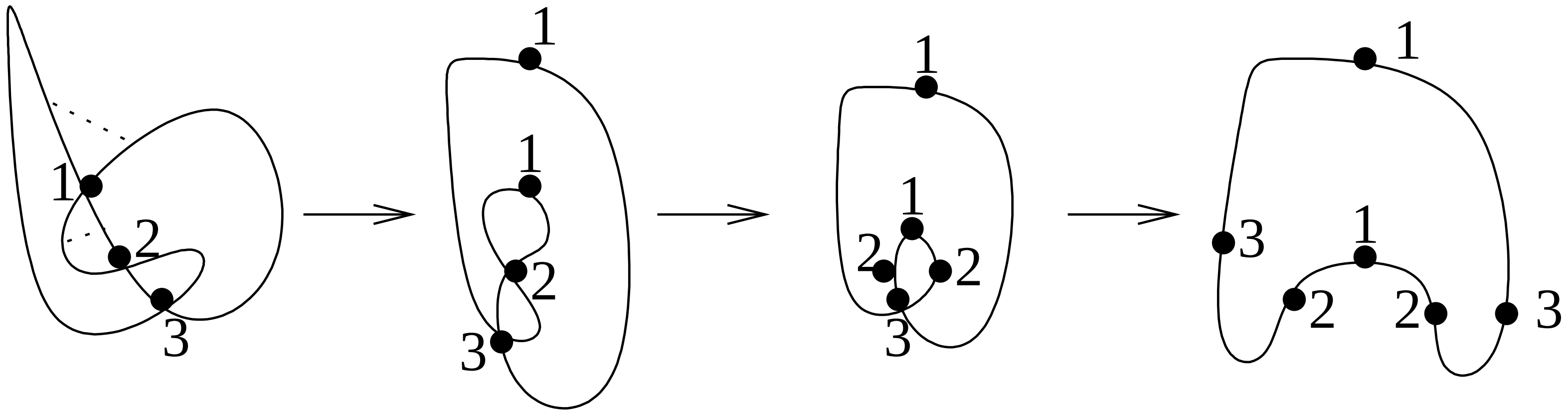

FIG. 3 – *Switch* of the word 123123

An important result

THEOREM [Dehn - 1936] An even word is realizable iff one can write the letter of its *Switch* on a circle and join the occurrences of the same letter without any crossing.

$\boxed{\text{An even word } A \text{ of } G_n \text{ is realizable iff } Switch(A) \text{ is a 2-player word}}$

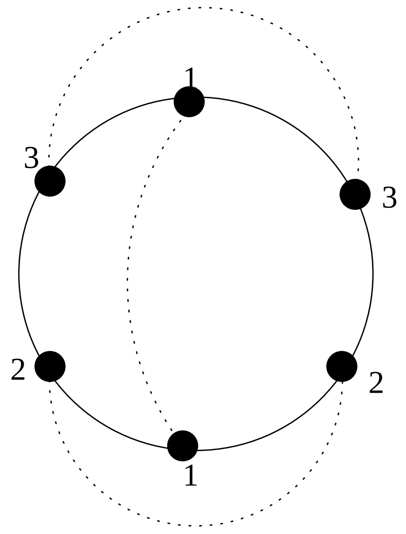

FIG. 4 – Player 1 takes 2, 3 and Player 2 takes 1

Still the same example !

$Switch(123123) = 132123$. And a game :

WORD	PLAYER1	PLAYER 2
132123		
13212	**3**	
1321	**23**	
132	23	**1**
13	$\cancel{2}$ $\cancel{2}$ 3	1
1	$\cancel{3}$ $\cancel{3}$	1
		$\cancel{1}$ $\cancel{1}$

So $A = 123123$ is realizable[a] .

[a]We already knew it !

$\boxed{\textbf{A first algorithm for realizability}}$

- Check the all 2^n games on $Switch(A)$.
- Complexity : $O(n.2^n)$.

Another consideration

- When $i - j - i - j$, A is said to be "alternating" .
- Then i and j can't be assigned to the same player.
- Consequently, $A = 1231245345$ is not realizable ($Switch(A) = 1354212345$; 1 is alternating with all other letters, but 3 is alternating with 4).

A polynomial algorithm

Systems

Let $A \in G_n$. Let x_1, x_2, x_n be formal variables : the equational system of A is the least set S containing the relations $x_i \neq x_k$ for i,j alternating in A, and such that the following holds for all i,j, k :

1. $(x_i \neq x_j) \in S \Rightarrow (x_j \neq x_i) \in S$
2. $(x_i = x_j) \in S \Rightarrow (x_j = x_i) \in S$
3. $(x_i = x_j) \in S$ and $(x_j = x_k) \in S \Rightarrow (x_i = x_k) \in S$
4. $(x_i = x_j) \in S$ and $(x_j \neq x_k) \in S \Rightarrow (x_i \neq x_k) \in S$
5. $(x_i \neq x_j) \in S$ and $(x_j = x_k) \in S \Rightarrow (x_i \neq x_k) \in S$
6. $(x_i \neq x_j) \in S$ and $(x_j \neq x_k) \in S \Rightarrow (x_i = x_k) \in S$

Consistency and realizability

The word A is "consistent" if the set S does not contain any contradiction, i.e $\forall i, (x_i \neq x_i) \notin S$.

$\boxed{\text{A word } A \text{ of } G_n \text{ is a 2-player word iff } A \text{ is consistent}}$

Proof :

$\Rightarrow$ if there exists a successful game on $Switch(A)$: denote by p_i the number of the player who took the letter i during this game . Then $x_i := p_i$ is a model of S.

$\Leftarrow$ the relation 6 forces any model of S to be isomorphic to $\{1, 2\}^n$.

The Final algorithm

A polynomial algorithm !

$$
\begin{array}{c}
A \\
{\scriptstyle O(n^2)} \downarrow \\
Switch(A) \\
{\scriptstyle O(n^3)} \downarrow \\
\text{Equational System of } Switch(A) \\
{\scriptstyle O(n^5)} \downarrow \\
\text{Minimal closure of the system and consistency}
\end{array}
$$

Perspectives

- The number of realizable Gauss words for $n = 3..12$:

n	3	4	5	6	7	8	9	10	11	12
#	1	1	2	3	10	27	101	364	1610	7202

- A systematic way to complete realizable words in order to obtain big ones ?